MATH206A MULTIVARIABLE CALCULUS - PROF. P. WONG

FINAL EXAM - DECEMBER 14, 2006

NAME:

Instruction: Read each question carefully. Explain **ALL** your work and give reasons to support your answers.

Advice: DON'T spend too much time on a single problem.

Problems	**Maximum Score**	**Your Score**
1.	15	
2.	20	
3.	15	
4.	20	
5.	15	
6.	15	
7.	20	
Total	120	

1. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$f(u, v) = (3u, -v).$$

(5 pts) (i) Find a matrix B so that $f(u, v) = B \begin{bmatrix} u \\ v \end{bmatrix}$.

(5 pts.) (ii) If D is the unit square $[0, 1] \times [0, 1]$, describe the image $D^* = f(D)$.

(5 pts.) (iii) Compute

$$\int\int_{D^*} x + y \, dA$$

by transforming the double integral into a double integral over D.

2. (10 pts) (i) Use change of variables to evaluate the following double integral. [Hint: First determine the region of integration.]

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}} e^{x^2+y^2}\ dxdy$$

(10 pts) (ii) Find the area of the region R bounded by the parabola $y = x^2 - 2$ and the line $y = x$. [Hint: Set up the double integral with an appropriate order of integration.]

3. Consider the following function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x, y) = x^5 + 2xy + y^3.$$

(8 pts) (i) Find a direction (give a unit vector) in which f increases most rapidly at the point $(2, -2)$.

(7 pts) (ii) Find an equation of the line tangent to the level curve $f(x, y) = 16$ at the point $(2, -2)$.

4. (8 pts) (i) Let C be the path formed by the square with vertices $(0,0),(1,0),(0,1)$, and $(1,1)$, oriented counterclockwise. Use Green's theorem to evaluate the line integral

$$\oint_C y^2\, dx + x^2\, dy.$$

(12 pts) (ii) Let $F(x,y) = (2x\sin y, x^2\cos y)$. Determine whether the vector field F is path independent. If F is path independent, find a function f so that $\nabla f = F$.

5. Consider the parametrized surface

$$f(s,t) = (s^2 - t^2, s + t, s^2 + 3t).$$

(8 pts) (i) Find a vector normal to this surface at the point $f(2,-1) = (3,1,1)$.

(7 pts.) (ii) Find an equation for the plane tangent to this surface at the point $(3,1,1)$.

6. Let $F(x,y,z) = (3x - xy, xy - yz, z(y-x) + \frac{z^2}{2})$.
(5 pts) (i) Find the divergence div F of F.

(10 pts) (ii) Use Gauss' (or Divergence) theorem to evaluate the surface integral

$$\oiint_{S^2} F \cdot \mathbf{n} \, d\sigma$$

where S^2 is the 2-dimensional unit sphere. [The volume of a ball of radius R is $\frac{4}{3}\pi R^3$.]

7. Let $F(x,y,z) = (z,x,y)$. Consider the parametrized surface M given by

$$f(s,t) = (s\cos t, s\sin t, t) \quad \text{where } 0 \le s \le 1, 0 \le t \le \frac{\pi}{2}.$$

(5 pts.)(i) Find curl F.

(5 pts.)(ii) Rewrite (not evaluate) the surface integral

$$\int\int_M \text{curl } F \cdot \mathbf{n} \, d\sigma$$

as a double integral over the region $R = \{(s,t)|0 \le s \le 1, 0 \le t \le \frac{\pi}{2}\}$.

(10 pts.)(iii) Use Stokes' theorem to evaluate the path integral

$$\oint_{\partial M} F \cdot d\mathbf{x} \quad \text{[Hint: Use part (ii).].}$$