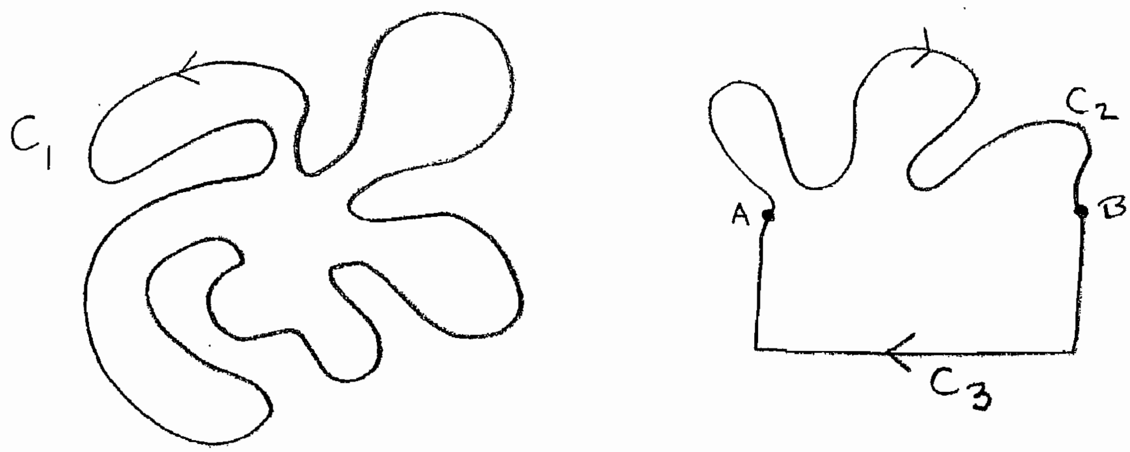
Name: Solutions

Exam 3- In-Class Portion

Show all your work to receive full credit for a problem.

1. Suppose $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ is a path independent vector field. Suppose C_1, C_2, C_3 are paths in $\mathbb{R}^2$ as pictured below.



(a) (3 pts) Can you compute $\oint_{C_1} \mathbf{F} \cdot d\mathbf{x}$? If so, what is it? If not, why not?

$\oint_{C_1} \vec{F} \cdot d\vec{x} = 0$, since $\vec{F}$ is path independent and C_1 is a closed curve.

(b) (3 pts) Is $\int_{C_2} \mathbf{F} \cdot d\mathbf{x} = \int_{C_3} \mathbf{F} \cdot d\mathbf{x}$? Explain why or why not.

They are not the same. By the fundamental theorem for path integrals, and since $\vec{F}$ is path independent,

$$\int_{C_2} \vec{F} \cdot d\vec{x} = f(B) - f(A)$$

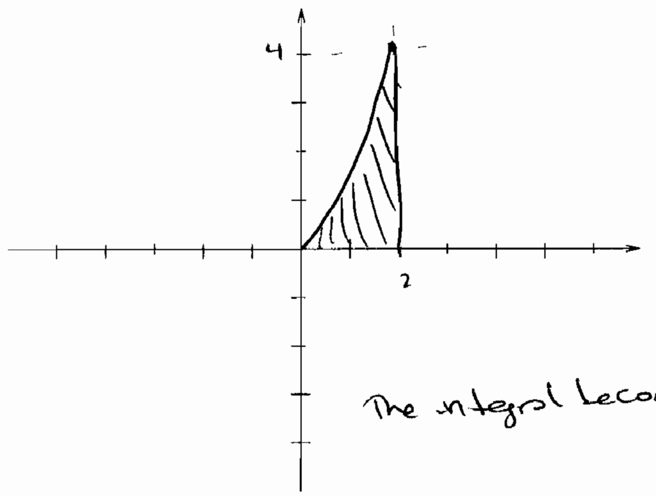
where f is such that $\vec{\nabla} f = \vec{F}$. But we also know

$$\int_{C_3} \vec{F} \cdot d\vec{x} = f(A) - f(B) = -\int_{C_2} \vec{F} \cdot d\vec{x}$$

So they differ in sign.

2. (6 pts) Sketch the region of integration in the coordinate axes provided and reverse the order of integration for the following integral:

$$\int_0^4 \int_{\sqrt{y}}^2 (x+y)\,dx\,dy$$



The region for this double integral is given by

$\sqrt{y} \le x \le 2$

$0 \le y \le 4$

which corresponds to the picture on the left. Switching to a y-simple region, we set

$0 \le x \le 2$

$0 \le y \le x^2$

The integral becomes

$$\int_0^2 \int_0^{x^2} (x+y)\,dy\,dx$$

3. (6 pts) Consider the path independent vector field $\mathbf{F} = (2xy + \cos 2y, x^2 - 2x \sin 2y)$. Find a potential field for $\mathbf{F}$.

We want to find $f : \mathbb{R}^2 \to \mathbb{R}$ such that $\vec{\nabla} f = \vec{F}$.

So we get the following equations

$$\frac{\partial f}{\partial x} = 2xy + \cos 2y$$

$$\frac{\partial f}{\partial y} = x^2 - 2x \sin 2y$$

Integrating on both sides of the first equation (with respect to x)

$$f(x,y) = \int \frac{\partial f}{\partial x}\,dx = \int 2xy + \cos 2y\,dx = x^2 y + x\cos 2y + K(y)$$

where $K(y)$ is a function of y. Differentiating this last equation with respect to y, we get

$$\frac{\partial f}{\partial y} = x^2 - 2x\sin 2y = x^2 - 2x\sin 2y + K'(y)$$

which means $K'(y) = 0$ so $K(y) = c$ some constant

Thus,

$$\boxed{f(x,y) = x^2 y + x\cos 2y + c}$$

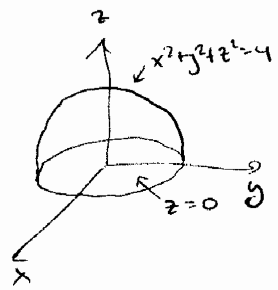
→ Any constant is allowed here, but this is the most general form.

4. Let S be the solid bounded above by the sphere $x^2 + y^2 + z^2 = 4$ and below by the plane $z = 0$, and suppose the density of S is given by the function

$$f(x, y, z) = \frac{1}{(x^2 + y^2 + z^2)^{3/2}}$$

(a) (6 pts) Write down an integral for the mass of S using rectangular coordinates. (DO NOT evaluate the integral).

The mass of a solid is given by $\iiint_S f \, dV$ where f is the density.



Since the solid is the top half of the sphere of radius 2, our limits of integration are easily described by assuming the solid is z-simple (there are other ways, but this is simplest).

$$0 \le z \le \sqrt{4-x^2-y^2}$$
$$-\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$$
$$-2 \le x \le 2$$

So

$$\text{Mass}(S) = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} \frac{1}{(x^2+y^2+z^2)^{3/2}} \, dz\, dy\, dx$$

(b) (6 pts) Set up an integral (but don't evaluate) in spherical coordinates which also gives you the mass of S.

The limits, described in spherical coordinates, would be

$0 \le \rho \le 2$ (the radius is at most 2)

$0 \le \theta \le 2\pi$ (all the ~~[illegible]~~ angles with respect to the x-axis are used)

(equiv.: the base is a circle)

$0 \le \phi \le \frac{\pi}{2}$ (we're only using the top half of the sphere)

We will also need the absolute value of the determinant of the Jacobian, which is $\rho^2 \sin\phi$, to change variables.

Finally, notice that $x^2+y^2+z^2 = \rho^2$. So

$$\text{Mass}(S) = \int_0^2 \int_0^{2\pi} \int_0^{\pi/2} \frac{1}{(\rho^2)^{3/2}} \cdot \rho^2 \sin\phi \, d\phi\, d\theta\, d\rho$$

$$= \int_0^2 \int_0^{2\pi} \int_0^{\pi/2} \frac{\sin\phi}{\rho} \, d\phi\, d\theta\, d\rho$$

5. (6 pts) Suppose you have a vector field $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$, defined by $\mathbf{F} = (F_1, F_2)$, such that

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0$$

Use Green's Theorem to show that $\mathbf{F}$ is path independent.

Green's theorem states that

$$\oint_{\partial R} \vec{F} \cdot d\vec{x} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\,dy$$

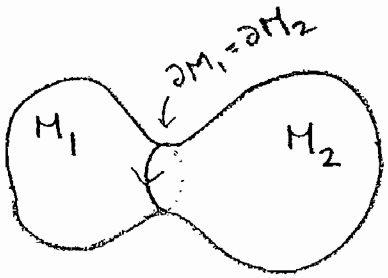
Let C be any (arbitrary) closed curve and assume it is the boundary of some region R. Then by Green's theorem,

$$\oint_C \vec{F} \cdot d\vec{x} = \iint_R 0\,dx\,dy = 0$$

Since C was arbitrary, we just showed that $\oint_C \vec{F} \cdot d\vec{x} = 0$ for all curves C, and by a theorem we know this means that $\vec{F}$ is path independent.

6. (6 pts) Suppose you have two surfaces, M_1, M_2 as pictured below. Explain why, for any continuously differentiable vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$,

$$\iint_{M_1} \text{curl}\mathbf{F} \cdot \mathbf{n} d\sigma = \iint_{M_2} \text{curl}\mathbf{F} \cdot \mathbf{n} d\sigma$$



By Stokes's theorem, we know that

$$\iint_{M_1} \text{curl}\vec{F} \cdot \vec{n}\, d\sigma = \oint_{\partial M_1} \vec{F} \cdot d\vec{x}$$

But $\partial M_1 = \partial M_2$, so we have

$$\oint_{\partial M_1} \vec{F} \cdot d\vec{x} = \oint_{\partial M_2} \vec{F} \cdot d\vec{x}$$

Now, using Stokes's theorem again, we see that

$$\oint_{\partial M_2} \vec{F} \cdot d\vec{x} = \iint_{M_2} \text{curl}\vec{F} \cdot \vec{n}\, d\sigma$$

And therefore we proved

$$\iint_{M_1} \text{curl}\vec{F} \cdot \vec{n}\, d\sigma = \iint_{M_2} \text{curl}\vec{F} \cdot \vec{n}\, d\sigma$$

TAKE-HOME SOLUTIONS

1 a) It is tempting to just use the parametrization and do this computation directly, but if you do that you realize quickly that the integral is very messy.

In general, it's easier to first check if $\vec{F}$ is path independent. For this, we would need

$$\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$

And since

$$\frac{\partial F_2}{\partial x} = \frac{-4xy}{(y^2+1)^2}$$

$$\frac{\partial F_1}{\partial y} = \frac{-2x}{(y^2+1)^2} \cdot 2y = \frac{-4xy}{(y^2+1)^2}$$

we know $\vec{F}$ is path independent.

C has endpoints at $(-1,0)$ and $(0,0)$ (Just plug in $t=0$ and $t=1$ into the parametrization)

So if we find f such that $\vec{\nabla} f = \vec{F}$, we can use the fundamental theorem for path integrals,

$$\int_C \frac{2x}{y^2+1}\,dx - \frac{2y(x^2+1)}{(y^2+1)^2}\,dy = f(0,0) - f(-1,0)$$

So we need to find f. This means we have to solve the two equations

$$\frac{\partial f}{\partial x} = \frac{2x}{y^2+1}$$

$$\frac{\partial f}{\partial y} = \frac{-2y(x^2+1)}{(y^2+1)^2}$$

Integrating on both sides of the first equation, we get

$$f(x,y) = \int \frac{2x}{y^2+1}\,dx$$

$$= \frac{x^2}{y^2+1} + K(y) \qquad \text{where } K \text{ is some function of } y.$$

Differentiating w/r to y

$$\frac{\partial f}{\partial y} = \frac{-x^2}{(y^2+1)^2}\cdot 2y + K'(y)$$

$$= \frac{-2x^2y}{(y^2+1)^2} + K'(y) \overset{\text{by assumption}}{=} -\frac{2y(x^2+1)}{(y^2+1)^2}$$

So $$K'(y) = \frac{-2y(x^2+1)}{(y^2+1)^2} + \frac{2x^2y}{(y^2+1)^2}$$

$$= \frac{-2y}{(y^2+1)^2}$$

Integrating with respect to y,

$$K(y) = \int \frac{-2y}{(y^2+1)^2}\,dy$$

$$= \frac{1}{y^2+1} + C \qquad \text{where } C \text{ is some constant.}$$

And so

$$f(x,y) = \frac{x^2}{y^2+1} + \frac{1}{y^2+1} + C$$

$$= \frac{x^2+1}{y^2+1} + C$$

Finally,

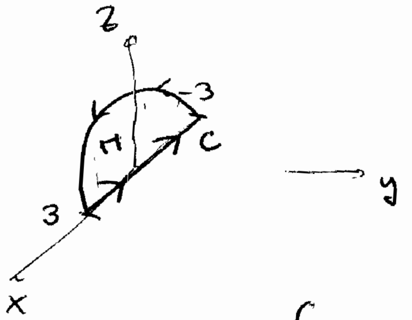
$$\int_C \vec{F}\cdot d\vec{x} = f(0,0) - f(-1,0)$$

$$= 1+C-(2+C)$$

$$= 1-2$$

$$= \boxed{-1}$$

1 b) The path we're integrating over looks like



We can use the definition directly, but I have chosen to solve this using Stokes's theorem. By stokes, we know

$$\oint_C (x+z)\,dx + x\,dy + y\,dz = \iint_M \operatorname{curl}\vec{F}\cdot\vec{n}\,d\sigma$$

$$\left(\begin{smallmatrix}\text{or}\\ \text{equivalently}\end{smallmatrix}\right) = \iint_M \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)dy\wedge dz + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)dz\wedge dx$$

$$+ \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dx\wedge dy$$

Since $y=0$ (our M lies on the xz-plane) we know $dy=0$, and therefore $dy\wedge dz = dx\wedge dy = 0$. So our integral is thus

$$= \iint_M \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)dz\wedge dx$$

$$= \iint_M (1-0)\,dz\wedge dx$$

$$= \iint_M dz\wedge dx$$

→ we want the normal to point in the positive y-direction, so we may assume $dz\wedge dx = dz\,dx$

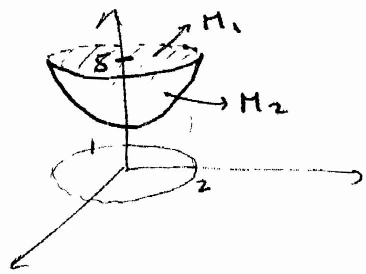
$$= \iint_M dz\,dx$$

= Area of the half circle of radius 3

$$= \frac{\pi(3)^2}{2}$$

$$= \boxed{\frac{9\pi}{2}}$$

2\. a) First, let's draw an approximate picture of our solid S. ④



Notice that the boundary of this solid is piecewise smooth, and the two smooth pieces that constitute the boundary are the paraboloid

$z = x^2+y^2+1$ where $x^2+y^2 \le 4$ (call this M_2)

and the disk at the top

$x^2+y^2 \le 4$, $z=5$. (call this M_1).

Therefore,

$$\sigma(\partial S) = \sigma(M_1) + \sigma(M_2).$$

The area of M_1 is easy to compute, since it's a disk of radius 2,

$$\sigma(M_1) = \pi(2)^2 = \boxed{4\pi}.$$

For the area of M_2, we need to find a parametrization. The most convenient one is

$$\vec{f} = (x, y, x^2+y^2+1), \quad x^2+y^2 \le 4.$$

So

$$\frac{\partial \vec{f}}{\partial x} = (1, 0, 2x)$$

$$\frac{\partial \vec{f}}{\partial y} = (0, 1, 2y)$$

And

$$\frac{\partial \vec{f}}{\partial x} \times \frac{\partial \vec{f}}{\partial y} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2x \\ 0 & 1 & 2y \end{vmatrix} = (-2x, -2y, 1)$$

$$\left\| \frac{\partial \vec{f}}{\partial x} \times \frac{\partial \vec{f}}{\partial y} \right\| = \sqrt{4x^2+4y^2+1}$$

And so now we can use our formula for surface area,

$$\sigma(M_2) = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{4x^2+4y^2+1}\, dy\, dx$$

This looks like a tricky integral, so we will see if polar coordinates help.

$x = r\cos\theta \quad 0 \le r \le 2$

$y = r\sin\theta \quad 0 \le \theta \le 2\pi$

$$\sigma(M_2) = \int_0^2 \int_0^{2\pi} \sqrt{4r^2+1} \cdot r\, d\theta\, dr = 2\pi \int_0^2 \sqrt{4r^2+1}\, r\, dr$$

using a u-substitution, ~~[illegible]~~

$u = 4r^2+1$

$du = 8r\,dr \rightarrow \frac{du}{8} = r\,dr$

$$= \frac{2\pi}{8} \int u^{1/2}\, du$$

$$= \frac{\pi}{4} \cdot \left(\frac{2}{3}\right) u^{3/2}$$

$$= \frac{\pi}{6} (4r^2+1)^{3/2} \Big|_0^2$$

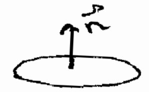
$$= \frac{\pi}{6} (17^{3/2} - 1)$$

So

$$\sigma(\partial S) = \sigma(M_1) + \sigma(M_2) = 4\pi + \frac{\pi}{6}(17^{3/2} - 1)$$

2b) Notice that since there are two pieces for the boundary, we need to compute

$$\iint_{\partial S} \vec{F} \cdot \vec{n}\, d\sigma = \iint_{M_1} \vec{F} \cdot \vec{n}\, d\sigma + \iint_{M_2} \vec{F} \cdot \vec{n}\, d\sigma$$

Let's first compute over M_1. Notice that we assumed ∂S was oriented outwards, so a suitable normal vector would be $\vec{n} = (0,0,1)$ ( → parallel to xy-plane) parametrization: $(x,y,5)$

$$\iint_{M_1} \vec{F}\cdot\vec{n}\, d\sigma = \iint_{M_1} (x,y,5)\cdot(0,0,1)\,d\sigma$$

$$= \iint_{M_1} 5\, d\sigma$$

$$= 5 \iint_{M_1} d\sigma$$

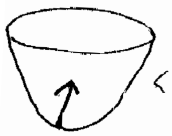
$= 5 \cdot$ Area of the circle of radius 2

$$= 5\cdot 4\pi$$

$$= \boxed{20\pi}$$

Now we need to compute the other surface integral. Notice that the parametrization we used in part (a) gave us

$\vec{n} = (-2x,-2y,1)$ which points up (so into the solid)



We can then choose $\vec{n} = (2x,2y,-1)$ so that we have the correct orientation. Or, we can say

$$\iint_{M_2} \vec{F}\cdot\vec{n}\,d\sigma = -\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (x,y,x^2+y^2+1)\cdot(-2x,-2y,1)\,dy\,dx$$

$$= -\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} -2x^2-2y^2+x^2+y^2+1\,dy\,dx$$

$$= -\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} -x^2-y^2+1\,dy\,dx$$

Again, it's convenient to switch to polar coordinates,

so $x = r\cos\theta$ $\quad 0 \le r \le 2 \quad 0 \le \theta \le 2\pi$

$y = r\sin\theta$

$$= -\int_0^2 \int_0^{2\pi} (-r^2+1) \cdot r \, d\theta \, dr$$

$$= -2\pi \int_0^2 -r^3 + r \, dr$$

$$= -2\pi \left(-\frac{r^4}{4} + \frac{r^2}{2} \right) \Big|_0^2$$

$$= -2\pi(-4+2)$$

$$= \boxed{4\pi}$$

So $\displaystyle\iint_{\partial S} \vec{F} \cdot \vec{n} \, d\sigma = 20\pi + 4\pi = \boxed{24\pi}$

c) Using the divergence theorem, we get that

$$\oiint_{\partial S} \vec{F} \cdot \vec{n} \, d\sigma = \iiint_S \operatorname{div} \vec{F} \, dV$$

The limits for the triple integral (in rectangular coordinates) are simply

$$x^2+y^2+1 \le z \le 5$$
$$-\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$$
$$-2 \le x \le 2$$

And $\operatorname{div} \vec{F} = \frac{\partial}{\partial x} x + \frac{\partial}{\partial y} y + \frac{\partial}{\partial z} z$

$$= 3$$

$$= -\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} -x^2 - y^2 + 1 \, dy\,dx$$

Polar $\quad x = r\cos\theta$

$\qquad\quad y = r\sin\theta$

$$= -\int_0^2 \int_0^{2\pi} (-r^2+1)\cdot r \, d\theta\, dr$$

$$= -2\pi \int_0^2 -r^3 + r \, dr$$

$$= -2\pi \left(-\frac{r^4}{4} + \frac{r^2}{2} \right)\Big|_0^2$$

$$= -2\pi \left(-\frac{16}{4} + \frac{4}{2} \right)$$

$$= -2\pi(-4+2)$$

$$= -2\pi(-2)$$

$$= \boxed{4\pi}$$

So $\iint_{\partial S} \vec{F}\cdot\vec{n}\, d\sigma = 20\pi + 4\pi = \boxed{24\pi}$

c) Using div theorem

$$\oiint_{\partial S} \vec{F}\cdot\vec{n}\, d\sigma = \iiint \operatorname{div}\vec{F}\, dV$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+y^2+1}^{5} 3 \, dz\,dy\,dx$$

$$= 3\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 5 - x^2 - y^2 - 1 \, dy\,dx$$

$$= 3\int_0^2\int_0^{2\pi} (4 - r^2) r \, d\theta\, dr$$

So

$$\iint_{\partial S} \vec{F} \cdot \vec{n}\, d\sigma = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+y^2+1}^{5} 3 \, dz\,dy\,dx$$

$$= 3 \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 5-x^2-y^2-1 \, dy\,dx$$

switch to cylindrical coordinates

$$= 3 \int_0^2 \int_0^{2\pi} (4-r^2) \cdot r \, d\theta\, dr$$

$$= 6\pi \int_0^2 4r - r^3 \, dr$$

$$= 6\pi \left(2r^2 - \frac{r^4}{4} \right) \Big|_0^2$$

$$= 6\pi (8-4)$$

$$= \boxed{24\pi}$$

which coincides with part (b) (as it should).

d) Since the answer is positive, we know that there is a flux of the field through the boundary going out of the solid (particles are flowing <u>out</u> of S)