 Name suggested solution

1. Consider the function $f(x, y) = 2x^3 - y^3 + 3x^2 - 36x + 3y$.

1a) Find ∇f.

$$\left(6x^2 + 6x - 36,\ -3y^2 + 3\right)$$

1b) Find all the critical points of f.

We need to solve the system $\begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{cases}$ $\begin{cases} 6x^2+6x-36 = 0 \Leftrightarrow x^2+x-6=0 \Leftrightarrow (x+3)(x-2)=0 \Leftrightarrow x=-3 \text{ or } x=2 \\ -3y^2+3=0 \Rightarrow y^2-1=0 \Leftrightarrow y=\pm 1 \end{cases}$

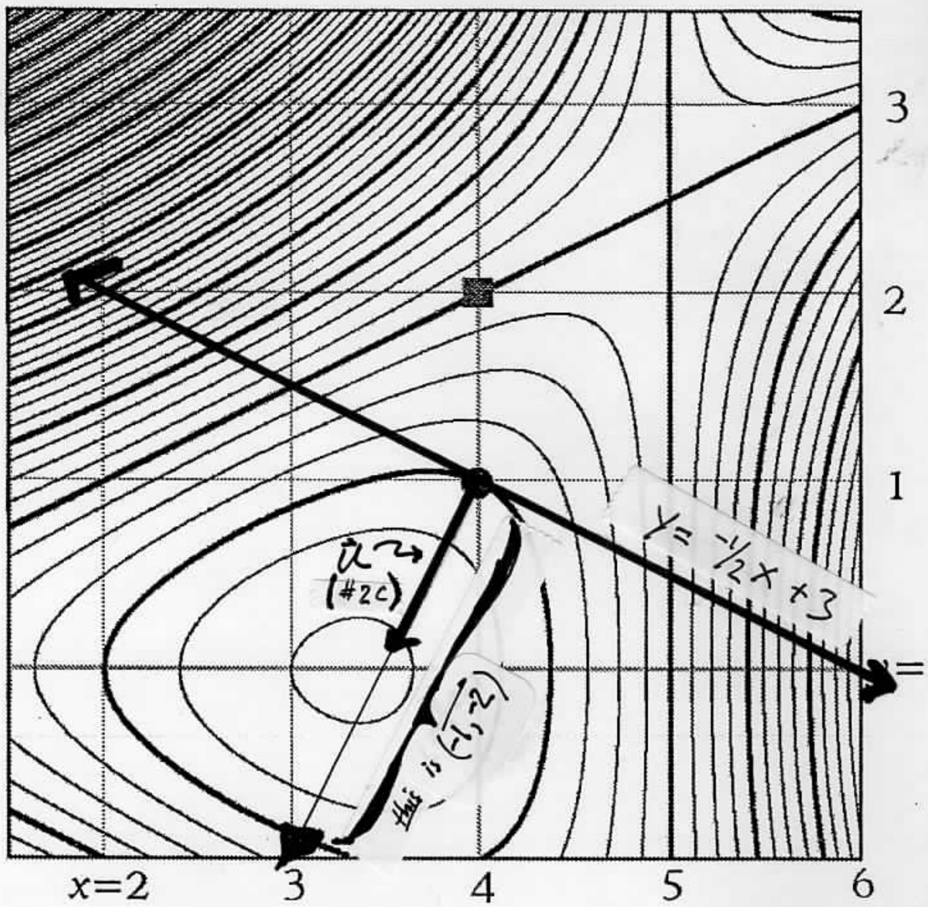
$\therefore$ $\boxed{\text{the complete list is } (-3,-1)\ (-3,1)\ (2\ -1)\ (2,1)}$

1c) Classify each of the critical points in problem (1b) as local max/mins/saddle points according to the second derivative test. If the test fails, explain why. ORGANIZE your answers neatly.

$f_{xx} = 12x+6 \qquad f_{yy} = -6y \qquad f_{xy} = 0$

Point a	$f_{xx}(a)$	$f_{xy}(a)$	$f_{yy}(a)$	$(f_{xx}f_{yy} - f_{xy}^2)\rvert_a = D$	results
$(-3,-1)$	$-30 < 0$	0	6	$-180 < 0$	$\begin{bmatrix} - & \\ & - \end{bmatrix}$ saddle pt
$(-3,1)$	$-30 < 0$	0	-6	$180 > 0$	$\begin{bmatrix} - & \\ & + \end{bmatrix}$ local max
$(2,-1)$	$30 > 0$	0	6	$180 > 0$	$\begin{bmatrix} + & \\ & + \end{bmatrix}$ local min
$(2,1)$	$30 > 0$	0	-6	$-180 < 0$	$\begin{bmatrix} + & \\ & - \end{bmatrix}$ saddle pt

2. Let $f(x,y) = 5x^2 - x^3 - 20y^2 + 4xy^2$. The accompanying graph shows level curves at intervals of 3, with those at intervals of 12 drawn with heavy lines.



2A) Find ∇f.

$$\overrightarrow{\left(10x - 3x^2 + 4y^2,\ -40y + 8xy\right)}$$

2B) There's a dot at (4,1). In the form $y = mx + b$, find the equation of the line tangent to the level curve there. Show all your work. Then make an excellent sketch, with the right slope, of this line on your graph.

we have: $\left(\overrightarrow{(x,y) - (4,1)}\right) \bullet \nabla f\big|_{(4,1)} = 0$

$$\left(\overrightarrow{x-4,\ y-1}\right) \bullet \left(\overrightarrow{-4,-8}\right) = 0$$

$$-4(x-4) - 8(y-1) = 0$$

$$-4x + 16 - 8y + 8 = 0$$

$$\boxed{y = -\tfrac{1}{2}x + 3}$$

(see sketch)

2C) Find a *unit* vector in the direction of the gradient at the point (4,1). Accurately sketch this vector on the graph, with its "tail" at (4,1). the gradient at $(4,1)$ is $(\overrightarrow{-4,-8})$ which has length $\sqrt{4^2+8^2} = \sqrt{80} = 4\sqrt{5}$

the corresponding unit vector is $\frac{1}{4\sqrt{5}}(\overrightarrow{-4,-8}) = \frac{\sqrt{5}}{5}(\overrightarrow{-1,-2})$ CALL IT $\vec{u}$. In the above figure, an easy way to draw this is to make a vector of length 1 along the vector $(\overrightarrow{-1,-2})$

2D) There appears to be a local maximum at some point $(p, 0)$ on the x-axis to the right of $(3, 0)$. Use ∇f to find the value of p exactly. At $(p,0)$, the gradient is $(\overrightarrow{10p - 3p^2 + 0,\ -40\cdot 0 + 8\cdot p\cdot 0})$

$$= (\overrightarrow{10p - 3p^2,\ 0}) = (\overrightarrow{p(10-3p),\ 0}), \text{ which is } 0 \Leftrightarrow p = 0 \text{ or } \tfrac{10}{3}$$

Since $(\frac{10}{3}, 0)$ is to the right of $(3,0)$ we take $p = 10/3$

2E) What is the equation of the plane tangent to the graph of f at this local maximum (ie, at $(p, 0, f(p, 0))$)? (this is an easy question: think about what the plane must look like at a maximum)

At a max, the tangent plane is parallel to the xy plane, so has the form $z = C$ where C is the value of f at $(p,0) = (\frac{10}{3}, 0)$.

$$\text{so: } C = f(\tfrac{10}{3}, 0) = 5\cdot\left(\tfrac{10}{3}\right)^2 - \left(\tfrac{10}{3}\right)^3 - 20\cdot 0^2 + 4\cdot\tfrac{10}{3}\cdot 0^2$$

$$= 5\cdot\frac{100}{9} - \frac{1000}{27}$$

$$= \frac{1500 - 1000}{27} = \frac{500}{27}; \text{ finally } \boxed{z = \frac{500}{27}}$$

(this is a continuation of problem 2)

2F) At the point (4,2), marked with a black square (■), the level curve looks like a straight line. Find in the form $y = mx + b$ the equation of this level "curve" just by looking at the graph.

The slope of that line is $\frac{1}{2}$; so $y-2 = \frac{1}{2}(x-4)$ is the equation.

We get $y = \frac{x}{2}$

2G) What is the value of f on that level curve (ie, the line) in 2F?

Since $(0,0)$ is on that line, $f(4,2) = f(0,0) =$ just plain 0.

2H) BONUS: show analytically that all the points on the line in 2F are indeed on that level curve.

(just show for any point of the form $(x, \frac{x}{2})$ that $f(x, \frac{x}{2}) = 0$)

2I) (not a bonus) Find the directional derivative in the direction of the vector $(4,5)$ at the point $(4,1)$.

We need a unit vector in the direction of $(4,5)$, which is $\frac{1}{\sqrt{4^2+5^2}}\overrightarrow{(4,5)} = \frac{1}{\sqrt{41}}\overrightarrow{(4,5)}$

then the directional derivative @ $(4,1)$ is $\nabla f\Big|_{(4,1)} \bullet \frac{1}{\sqrt{41}}\overrightarrow{(4,5)} = \frac{1}{\sqrt{41}}\overrightarrow{(-4,-8)} \bullet \overrightarrow{(4,5)}$

$= \frac{1}{\sqrt{41}}(-16-40) = \frac{-56}{\sqrt{41}}$

2J) (not a bonus) Find the equation of the plane tangent to the graph of $f(x,y)$ at the point $(4,1,f(4,1))$

Oneway: Consider this graph to be the level surface of $g(x,y,z) = f(x,y) - z$ at $c=0$, with the point $(4,1,f(4,1)) = (4,1,12)$ being on this surface. So $g(x,y,z) = 5x^2 - x^3 - 20y^2 + 4xy^2 - z$

The gradient here is $\nabla g\Big|_{(4,1,12)} = \overrightarrow{(10x - 3x^2 + 4y^2, -40y + 8xy, -1)}\Big|_{(4,1,12)}$

$= \overrightarrow{(-4,-8,-1)}$

and it's $\perp$ to the tangent plane.

$\therefore$ A point (x,y,z) is on this plane $\Leftrightarrow$

$\overrightarrow{(x,y,z)-(4,1,12)} \bullet \overrightarrow{(-4,-8,-1)} = 0$

Simplify to get

$-4(x-4) - 8(y-1) - 1(z-12) = 0$

$-4x - 8y - z + 16 + 8 + 12 = 0$

$\boxed{4x + 8y + z - 36 = 0}$

another way: find the 1^{st}-degree Taylor poly. approx to f at $(4,1)$.

4A. Find the second order Taylor polynomial for $f(x,y) = y\sqrt{x}$ for the point $\mathbf{a} = (25, 3)$.

The second order Taylor polynomial is

$$TP_2 = f(a) + (\vec{h}\cdot\nabla)(f)\Big|_a + \frac{(\vec{h}\cdot\nabla)^2 f}{2!}\Big|_a \quad \text{where } \vec{h}\cdot\nabla \text{ is } \overrightarrow{(x-25, y-3)}\cdot(\partial/\partial x, \partial/\partial y)$$

$$= (x-25)\,\partial/\partial x + (y-3)\,\partial/\partial y$$

and so $(\vec{h}\cdot\nabla)^2$ is the operator above SQUARED, meaning

$$(x-25)^2\,\partial/\partial x(\partial/\partial x(\ )) + 2(x-25)(y-3)\,\partial/\partial x(\partial/\partial y(\ )) + (y-3)^2\,\partial/\partial y(\partial/\partial y(\ )),$$

and applying either operator to f and evaluating the result at a is what the "$f|_a$" means.

$$f(a) = 3\cdot\sqrt{25} = 15$$

$$f_x|_a = \partial/\partial x(yx^{1/2})\Big|_a = \tfrac{1}{2}yx^{-1/2}\Big|_{(25,3)} = \tfrac{1}{2}\cdot 3\cdot\tfrac{1}{5} = \tfrac{3}{10}$$

$$f_y|_a = \partial/\partial y(yx^{1/2})\Big|_a = x^{1/2}\Big|_{(25,3)} = 5$$

$$f_{xx}|_a = \partial/\partial x(\partial/\partial x(yx^{1/2}))\Big|_a = \partial/\partial x\left(\tfrac{1}{2}yx^{-1/2}\right)\Big|_a = \tfrac{-1}{4}yx^{-3/2}\Big|_a$$

$$= \tfrac{-1}{4}\cdot 3\cdot(25)^{-3/2} = \frac{-3}{4\cdot 125} = \frac{-3}{500}$$

$$f_{yx}|_a = \partial/\partial x(\partial/\partial y(yx^{1/2}))\Big|_a = \partial/\partial x(x^{1/2})\Big|_a = \tfrac{1}{2}x^{-1/2}\Big|_a = \tfrac{1}{2}\cdot\tfrac{1}{5} = \tfrac{1}{10}.$$

$$f_{yy}|_a = \partial/\partial y(\partial/\partial y(yx^{1/2}))\Big|_a = \partial/\partial y(x^{1/2})\Big|_a = 0|_a = 0$$

we get: $TP_2 = 15 + (x-25)\cdot\frac{3}{10} + (y-3)\cdot 5 + \dfrac{(x-25)^2\left(\frac{-3}{500}\right) + 2(x-25)(y-3)\cdot\frac{1}{10} + (y-3)^2\cdot 0}{2!}$

$$\boxed{= 15 + \tfrac{3}{10}(x-25) + 5(y-3) + \tfrac{-3}{1000}(x-25)^2 + \tfrac{1}{10}(x-25)(y-3)}$$

4B) Use that polynomial from 4A to approximate $3.02\sqrt{26}$ (that is, $f(26, 3.02)$). Compare the approximation to the actual (calculator) value by finding the differences.

$f(26, 3.02) \approx TP_2(26, 3.02)$. Now, at $(26, 3.02)$, $x-25$ becomes $26-25=1$ and $y-3 = 3.02-3 = 0.02$

so the TP_2 becomes (at $(26, 3.02)$):

$$15 + \frac{3}{10} + 5(0.02) + \frac{-3}{1000} + \frac{1}{10}(0.02)$$

$$= 15 + 0.3 + 0.1 - 0.003 + 0.002$$

$$= 15.4 - 0.001 = 15.399 \quad \text{(no calculator required!)}$$

whereas $f(26, 3.02)$ by calculator is $15.39903893\ldots$ a difference of $\underline{0.00003893}$

5. Let **F** be the vector field $\mathbf{F}(x, y, z) = (x^2yz^3,\ 5x + 4y + 3z,\ x^2 + y^2z^2)$ and find both the divergence and curl of this vector field.

5A: the divergence is:

$$\operatorname{div} \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 2xyz^3 + 4 + 2y^2z$$

5B the curl is: the vector represented by

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2yz^3 & 5x+4y+3z & x^2+y^2z^2 \end{vmatrix}$$

$$= (2yz^2 - 3)\mathbf{i} + (3x^2yz^2 - 2x)\mathbf{j} + (5 - x^2z^3)\mathbf{k}$$

 Name suggested solution

6A. Suppose $\mathbf{g}(r,p,s)$ has two component functions g_1 and g_2. Suppose each of r, p and s is a function of x and y; let $\mathbf{f}(x,y)$ have r, p and s as its component functions. With this much information, find (the entries of) both $J\mathbf{g}$ and $J\mathbf{f}$. (for example, one of the entries in $J\mathbf{g}$ is $\partial g_2/\partial p$; one of the entries in $J\mathbf{f}$ is $\partial p/\partial y$).

$$J\vec{g} = \begin{bmatrix} \frac{\partial g_1}{\partial r} & \frac{\partial g_1}{\partial p} & \frac{\partial g_1}{\partial s} \\ \frac{\partial g_2}{\partial r} & \frac{\partial g_2}{\partial p} & \frac{\partial g_2}{\partial s} \end{bmatrix} ; \quad J\vec{f} = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial p}{\partial x} & \frac{\partial p}{\partial y} \\ \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \end{bmatrix}$$

6B. Let $\mathbf{u} = \mathbf{g} \circ \mathbf{f}$, then $\mathbf{u}$ is a function of x and y. Let its component functions be u_1, u_2, etc. Use the chain rule and the Jacobians in 6A to find an expression for $\partial u_2/\partial y$.

$$D\vec{u} = D\vec{g}\, D\vec{f} = \begin{bmatrix} * & * \\ * & \frac{\partial u_2}{\partial y} \end{bmatrix}$$

$$= \text{" } 2^{nd} \text{ row of } J\vec{g} \cdot 2^{nd} \text{ col of } J\vec{f} \text{ "}$$

$$= \boxed{\frac{\partial g_2}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial g_2}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial g_2}{\partial s}\frac{\partial s}{\partial y}}$$