MATH206A MULTIVARIABLE CALCULUS - PROF. P. WONG

EXAM I - SEPTEMBER 29, 2006

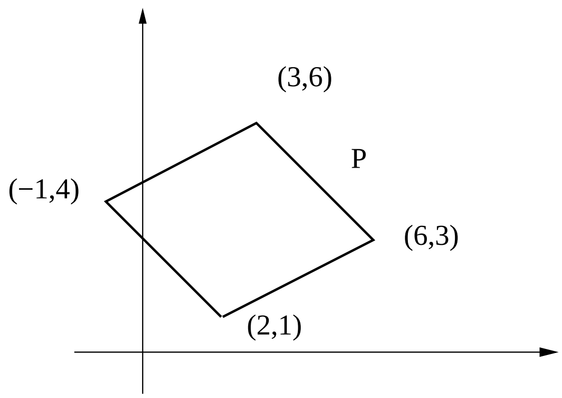
NAME:

Instruction: Read each question carefully. Explain **ALL** your work and give reasons to support your answers.

Advice: DON'T spend too much time on a single problem.

Problems	**Maximum Score**	**Your Score**
1.	20	
2.	20	
3.	20	
4.	20	
5.	20	
Total	100	

1. Let P be the parallelogram in $\mathbb{R}^3$ with vertices $(2,1), (-1,4), (6,3)$ and $(3,6)$.



(5 pts) (i) Find the area of P.

Let $\mathbf{a} = (-1,4) - (2,1) = (-3,3)$ *and* $\mathbf{b} = (6,3) - (2,1) = (4,2)$. *Then the area of* P *is equal to* $||\mathbf{a} \times \mathbf{b}||$*=18.*

(3 pts) (ii) Suppose $T(x_1, x_2) = (5x_1 + 4x_2, 5x_1 + 3x_2)$. Find the associated matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ where $\mathbf{x} = (x_1, x_2)$.

The associated matrix is

$$A = \begin{pmatrix} 5 & 4 \\ 5 & 3 \end{pmatrix}$$

(4 pts) (iii) What are the vertices of the image $T(P)$?

The vertices of $T(P)$ *are* $(11,7), (14,13), (42,39)$, *and* $(39,33)$.

(4 pts) (iv) What is the area of $T(P)$?

Area of $T(P)$ *is given by*

$$|\det A| \cdot \textit{area of } P = |(5)(3) - (5)(4)| \cdot 18 = 90.$$

(4 pts) (v) What is the angle of the parallelogram P at the vertex $(6,3)$? (You may express it in terms of inverse trig function.)

Since $||\mathbf{a} \times \mathbf{b}|| = 18 = (||\mathbf{a}||)(||\mathbf{b}||)\sin\theta$. *It follows that* $\theta = \arcsin\left(\frac{3}{\sqrt{10}}\right)$.

2. Let $\mathbf{a} = \mathbf{i} + \mathbf{j} - 6\mathbf{k}$, $\mathbf{b} = 4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$, and $\mathbf{c} = -\mathbf{i} + 2\mathbf{k}$.

(7 pts) (i) Find a vector that is perpendicular to the plane spanned by $\mathbf{a}$ and $\mathbf{b}$.

The cross product $\mathbf{a} \times \mathbf{b}$ *is orthogonal to the plane containing* $\mathbf{a}$ *and* $\mathbf{b}$*. This vector is given by*

$$\mathbf{a} \times \mathbf{b} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -6 \\ 4 & 3 & 1 \end{pmatrix} = 19\mathbf{i} - 25\mathbf{j} - \mathbf{k}.$$

(7 pts) (ii) What is the volume of the parallelepiped formed by $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$?

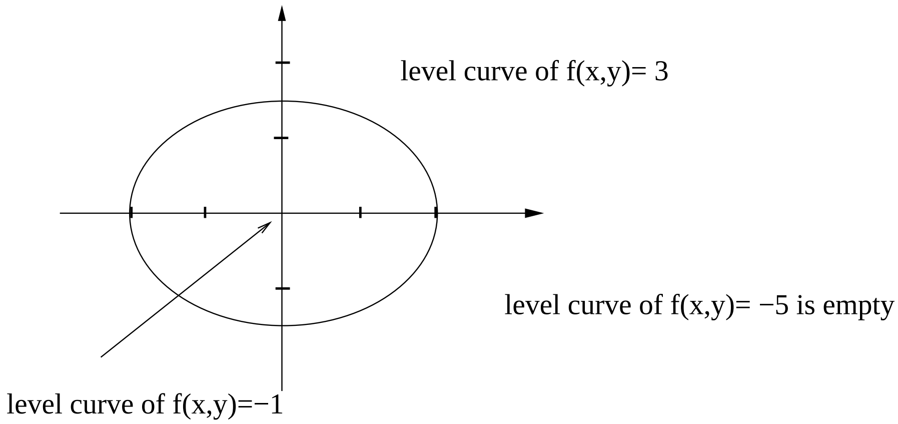
The volume is given by $|\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})| = |(-1, 0, 2) \cdot (19, -25, -1)| = 21.$

(6 pts) (iii) Find $\text{proj}_{\mathbf{c}}\mathbf{a}$.

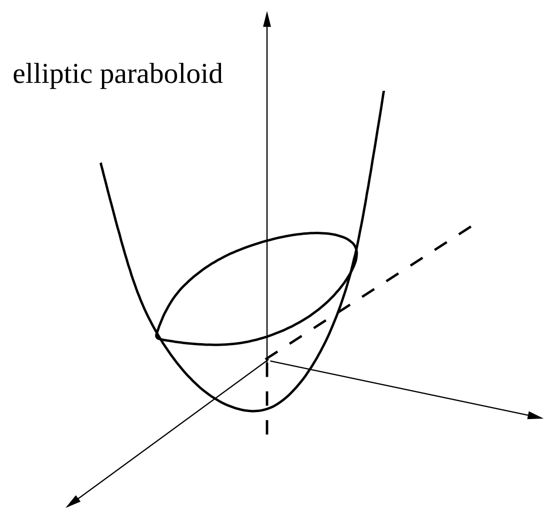
Since $\mathbf{a} \cdot \mathbf{c} = (1)(-1) + (1)(0) + (-6)(2) = -13$ *and* $||\mathbf{c}||^2 = (-1)^2 + (2)^2 = 5$, *it follows that*

$$\begin{aligned} \text{proj}_{\mathbf{c}}\mathbf{a} &= \frac{\mathbf{a} \cdot \mathbf{c}}{||\mathbf{c}||^2}\mathbf{c} \\ &= \frac{13}{5}\mathbf{i} - \frac{26}{5}\mathbf{k}. \end{aligned}$$

3. Let $f(x, y) = x^2+2y^2-1$. (5 pts) (i) Sketch the level curves of $f(x, y) = 3$, $f(x, y) = -1$, and $f(x, y) = -5$.



(5 pts) (ii)Describe or sketch the set of points in $\mathbb{R}^3$ that satisfy the equation $f(x, y) = x^2 + 2y^2 - 1$ (or the graph of $z = f(x, y)$)



(5 pts) (iii)What are the cylindrical coordinates of the point $(1, 2, 8)$?

In cylindrical coordinates, we have $r = \sqrt{x^2 + y^2} = \sqrt{5}$ *and* $\tan\theta = y/x$ *so that* $\theta = \arctan 2$. *The point* $(1, 2, 8)$ *becomes* $(\sqrt{5}, \arctan 2, 8)$ *in cylindrical coordinates.*

(5 pts) (iv) Write the equation $z = x^2 + 2y^2 - 1$ in spherical coordinates.

In spherical coordinates, $x = \rho\sin\phi\cos\theta, y = \rho\sin\phi\sin\theta$, *and* $z = \rho\cos\phi$. *Thus the equation in spherical coordinates becomes* $\rho\cos\phi = (\rho\sin\phi)^2\cos^2\theta + 2\rho^2\sin^2\phi\sin^2\theta - 1$.

4. The velocity of a particle in $\mathbb{R}^3$ is given by the parametrization

$$v(t) = \mathbf{i} + (1+t)\mathbf{j} + \cos t\mathbf{k}.$$

(10 pts) (i) Find the position $r(t)$ of the particle with the initial point $r(0) = \mathbf{i} + \mathbf{k}$.

Note that

$$\begin{aligned} r(t) &= \int v(t)\ dt \\ &= (t + C_1)\mathbf{i} + (t + \frac{t^2}{2} + C_2)\mathbf{j} + (\sin t + C_3)\mathbf{k} \end{aligned}$$

where C_1, C_2, *and* C_3 *are constants. Since* $r(0) = \mathbf{i} + \mathbf{k}$, *it follows that* $C_1 = 1, C_2 = 0$, *and* $C_3 = 1$. *Hence,*

$$r(t) = (t+1)\mathbf{i} + (t + \frac{t^2}{2})\mathbf{j} + (\sin t + 1)\mathbf{k}.$$

(10 pts) (ii) Give an equation (in vector form) of the line tangent to the path of the particle at the point $r(\pi)$.

The vector $v(\pi) = \mathbf{i} + (1+\pi)\mathbf{j} - \mathbf{k}$ *is in the same direction as the desired tangent line. Hence, the tangent line to the path at* $r(\pi)$ *is given by*

$$\begin{aligned} \mathbf{x}(t) &= r(\pi) + tv(\pi) \\ &= (\pi + 1)\mathbf{i} + (\pi + \frac{\pi^2}{2})\mathbf{j} + \mathbf{k} + t[\mathbf{i} + (1+\pi)\mathbf{j} - \mathbf{k}] \end{aligned}$$

5. (10 pts) (i) Consider the function

$$f(x, y) = \frac{xy}{x^2 + y^2}.$$

Determine whether the origin is a removable discontinuity of f. Justify your answer. [Hint: try approaching $(0, 0)$ from different directions]

To determine whether $(0, 0)$ *is a removable discontinuity of* f*, it suffices to determine whether the limit* $\lim_{(x,y)\to(0,0)} f(x, y)$ *exists. Suppose we let* (x, y) *approach to* $(0, 0)$ *along the line* $y = mx$ *for some* $m \neq 0$*. For any such* m*,* $f(x, y) = \frac{mx^2}{(1+m^2)x^2}$*. Thus for* $m \neq 0$*,*

$$\lim_{(x,y)\to(0,0)} f(x, y) = \frac{m}{1 + m^2}$$

which clearly varies as m *varies so the limit does not exist and hence* $(0, 0)$ *is not removable.*

(10 pts) (ii) Give an equation for the plane that is perpendicular to the line with parametric equations $x = 3t - 5, y = 7 - 2t, z = 8 - t$ and that contains the point $(1, -1, 2)$. [Hint: rewrite the line in vector form $\mathbf{x} = t\mathbf{m} + \mathbf{x}_0$]

In vector form, the given line is $\mathbf{x} = t\mathbf{m} + \mathbf{x}_0$ *where* $\mathbf{m} = (3, -2, -1)$ *and* $\mathbf{x}_0 = (-5, 7, 8)$*. This means that the vector* $(3, -2, -1)$ *is a vector perpendicular to the desired plane, which contains the point* $(1, -1, 2)$*. If an arbitrary point on this plane is denoted by* (x, y, z) *then the vector* $(x-1, y+1, z-2)$ *must be perpendicular to the vector* $(3, -2, -1)$*, in other words, the desired plane is given by*

$$(x - 1, y + 1, z - 2) \cdot (3, -2, -1) = 0 \quad \textit{or} \quad 3x - 2y - z = 3.$$