MATH206A MULTIVARIABLE CALCULUS - PROF. P. WONG

EXAM I - SEPTEMBER 27, 2007

NAME:

Instruction: Read each question carefully. Explain **ALL** your work and give reasons to support your answers.

Advice: DON'T spend too much time on a single problem.

Problems	**Maximum Score**	**Your Score**
1.	16	
2.	17	
3.	17	
4.	17	
5.	16	
6.	17	
Total	100	

1. Let P be the parallelogram in $\mathbb{R}^3$ with vertices

$$A = (1, -1, 2), B = (2, 0, 1), C = (3, 2, -1), \text{ and } D = (2, 1, 0).$$

[Don't spend too much time drawing the picture!]
(8 pts) (i) Find the area of P.

The vector $\vec{AB}$ **is** $(1, 1, -1)$ **and** $\vec{DC} = (1, 1, -1)$ **so** AB **is parallel to** CD**. In other words,** AB **and** AD **are two adjacent edges of the parallelogram at the corner** A**. Thus the area of** P **is given by**

$$\begin{aligned} area(P) &= \|\vec{AB} \times \vec{AD}\| \\ &= \left\| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ 1 & 2 & -2 \end{vmatrix} \right\| \\ &= \|\mathbf{j} + \mathbf{k}\| = \sqrt{2}. \end{aligned}$$

(8 pts) (ii) Let $E = (2, -2, 5)$. Find the volume of the parallelepiped spanned by the vectors $\vec{AB}, \vec{AD}$ and $\vec{AE}$.

The vector $\vec{AE}$ **is given by** $(1, -1, 3)$**. From (i), we have** $\vec{AB} \times \vec{AD} = \mathbf{j} + \mathbf{k}$**. It follows that the volume of the desired parallelepiped is given by**

$$|\vec{AE} \cdot (\vec{AB} \times \vec{AD})| = |(1)(0) + (-1)(1) + (3)(1)| = 2.$$

2. Consider the following two planes $P_1 : 4x - y + z = 2$ and $P_2 : 2x - z = 3$ in $\mathbb{R}^3$.

(5 pts) (i) Find a point on the line of intersection between the planes P_1 and P_2.

There are infinitely many points that lie on the line of intersection. In particular, when $z = 0$**, the line** $4x - y = 2$ **lies on** P_1**; the line** $2x = 3$ **lies on** P_2 **and thus** $x = \frac{3}{2}$**. On the line** $4x - y = 2$**,** $y = 4$**. Hence, the point** $(\frac{3}{2}, 4, 0)$ **lies on both** P_1 **and** P_2**.**

(3 pts) (ii) Find a vector orthogonal to the plane P_1.

Since $4x - y + z = 2$**, we have** $4(x - 0) + (-1)(y - 0) + 1(z - 2) = 0$ **so that** $(4, -1, 1)$ **is orthogonal to** P_1**.**

(3 pts) (iii) Find a vector orthogonal to the plane P_2.

Similar to (ii), $(2, 0, -1)$ **is orthogonal to** P_2**.**

(6 pts) (iv) Find a parametrization for the line of intersection between P_1 and P_2.

From (i), $\mathbf{x_0} = (\frac{3}{2}, 4, 0)$ **is on the line of intersection. The vector**

$$(4, -1, 1) \times (2, 0, -1) = (1, 6, 2)$$

is in the same direction as the line of intersection. Thus,

$$\mathbf{x} = (\frac{3}{2}, 4, 0) + t(1, 6, 2)$$

is a parametrization of the desired line.

3. The position of a particle in $\mathbb{R}^3$ is given by the parametrization

$$r(t) = e^t\mathbf{i} + \mathbf{j} + \sin t\mathbf{k}, \quad \text{for } t \geq 0.$$

(5 pts) (i) Find the velocity $v(t)$ of the particle at time t.

$$v(t) = r'(t) = e^t\mathbf{i} + 0\mathbf{j} + \cos t\mathbf{k}.$$

(6 pts) (ii) Find the projection of the initial velocity in the direction of $\mathbf{a} = \mathbf{i} + 2\mathbf{j}$.

Initial velocity $v(0) = \mathbf{i} + \mathbf{k}$**. Then**

$$\begin{aligned}\text{proj}_{\mathbf{a}} v(0) &= \frac{v(0) \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} \\ &= \frac{1}{5}(\mathbf{i} + 2\mathbf{j}).\end{aligned}$$

(6 pts) (iii) Find the angle between the position and the velocity of the particle at $t = \frac{\pi}{4}$.

First, $r\left(\frac{\pi}{4}\right) = (e^{\frac{\pi}{4}}, 1, \frac{\sqrt{2}}{2})$ **and** $v\left(\frac{\pi}{4}\right) = (e^{\frac{\pi}{4}}, 0, \frac{\sqrt{2}}{2})$**. It follows that**

$$r\left(\frac{\pi}{4}\right) \cdot v\left(\frac{\pi}{4}\right) = e^{\frac{\pi}{2}} + \frac{1}{2}.$$

On the other hand, the dot product above is equal to $\|r\|\|v\|\cos\theta$ **where** θ **is the angle between the position and the velocity. A straightforward computation shows that**

$$\theta = \arccos\left(\sqrt{\frac{e^{\frac{\pi}{2}} + \frac{1}{2}}{e^{\frac{\pi}{2}} + \frac{3}{2}}}\right).$$

4.(4 pts) (i) Consider the line given by the parametric equations $x = t + 1, y = 2 - 3t, z = 2t + 1$. Write this line in vector form $\mathbf{x} = t\mathbf{m} + \mathbf{x}_0$

$$\mathbf{x} = t(1, -3, 2) + (1, 2, 1).$$

(8 pts) (ii) Give an equation for the plane that is perpendicular to the line given in (i) and that contains the point $(1, 1, 0)$. [Hint: use (i)]

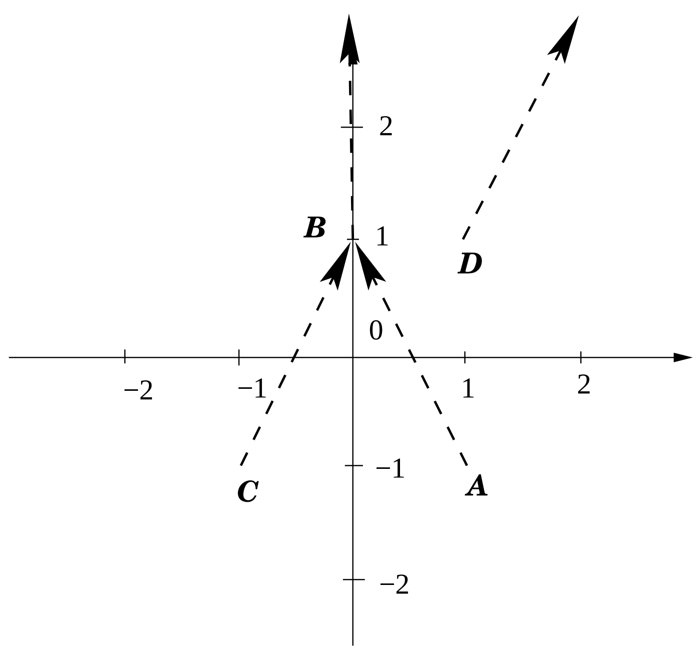
Since the point $(1, 1, 0)$ **is on the plane, the vector** $\mathbf{x} - (1, 1, 0)$ **is parallel to the plane where** $\mathbf{x} = (x, y, z)$ **denotes an arbitrary point on the plane. Moreover, the vector** $(1, -3, 2)$ **is orthogonal to this plane. Thus,**

$$(\mathbf{x} - (1, 1, 0)) \cdot (1, 3, 2) = 0$$

or equivalently,

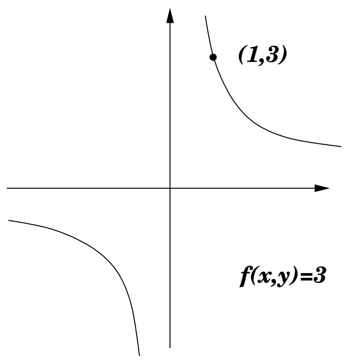
$$x - 3y + 2z + 2 = 0.$$

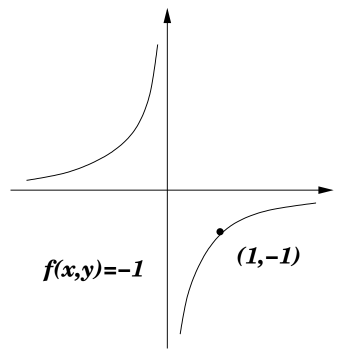
(5 pts) (iii) Sketch the vector field $f(x, y) = (xy, 2)$ at the points $A = (1, -1), B = (0, 1), C = (-1, -1), D = (1, 1)$.



5. Let $f(x, y) = xy$.

(5 pts) (i) Sketch the level curves of $f(x, y) = 3$ and $f(x, y) = -1$.





(5 pts) Describe or sketch the level set $h(x, y, z) = 4$ where $h(x, y, z) = x^2 + \frac{y^2}{9}$.

Since $h(x, y, z) = x^2 + \frac{y^2}{9}$**, the level set** $h(x, y, z) = 4$ **is simply** $x^2 + \frac{y^2}{9} = 4$**. Since this equation holds for** ANY **value of** z**, it follows that the level set (surface) is an infinite elliptic cylinder.**

(6 pts) (iii)What are the cylindrical coordinates (r, θ, w) **AND** the spherical coordinates (ρ, θ, ϕ) of the point $(1, 1, \sqrt{2})$?

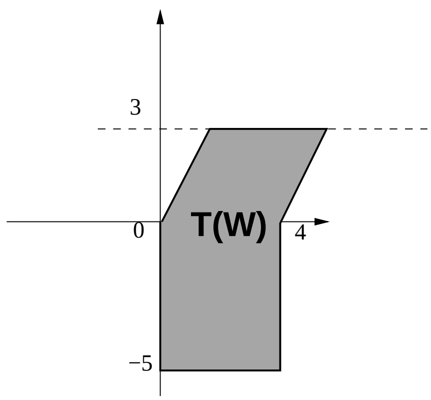
For cylindrical coordinates, we have $x = r\cos\theta, y = r\sin\theta, w = z$**. Note that** $r^2 = x^2 + y^2 = 1 + 1 = 2$ **so** $r = \sqrt{2}$**. Moreover,** $1 = \sqrt{2}\cos\theta$ **which implies that** $\theta = \frac{\pi}{4}$**. The cylindrical coordinates of the point** $(1, 1, \sqrt{2})$ **are** $(\sqrt{2}, \frac{\pi}{4}, \sqrt{2})$**. For the spherical coordinates,** $\rho^2 = x^2 + y^2 + z^2 = 4$ **or** $\rho = 2$**. Since** $\theta = \frac{\pi}{4}$ **and** $z = \rho\cos\phi = \sqrt{2}$**, we have** $\phi = \frac{\pi}{4}$**. The spherical coordinates of the point** $(1, 1, \sqrt{2})$ **are** $(2, \frac{\pi}{4}, \frac{\pi}{4})$**.**

6. Let T be a linear transformation given by $T(x_1, x_2) = (3x_2 - x_1, 2x_1)$.
(7 pts)(i) Find the matrix A associated to T such that $T(\mathbf{x}) = A\mathbf{x}$.

The matrix A associated with the linear transformation T is given by

$$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}.$$

(10 pts)(ii) Suppose W is a figure in $\mathbb{R}^2$ so that $T(W)$ is the figure shown below.



What is the area of the figure W? [Hint: $T(W)$ is made up of a parallelogram and a rectangle; what is the relation between the area of W and the area of $T(W)$?]

Since $T(W)$ is a union of two parallelgrams, then so is W. It follows that

$$area(T(W)) = |\det A| \cdot area(W).$$

Now, $T(W)$ is the sum of the two areas which is $(3)(4) + (4)(5) = 32$ and $\det A = -6$. Thus, $area(W) = \frac{32}{6}$.