1. **50 points** All parts below refer to the function \( f(x) = \frac{\sin x}{x} \).

   a. What is the average rate of change of \( f \) on \([2, 4]\)?

   \[
   \frac{f(4) - f(2)}{4 - 2} = \frac{\sin 4 - \sin 2}{2} \approx -0.322
   \]

   b. What is the average value of \( f \) on \([2, 4]\)?

   \[
   \frac{1}{4 - 2} \int_2^4 \frac{\sin x}{x} \, dx \approx \frac{153.8}{2} = 76.4
   \]

   c. What is the derivative of \( f \) at \( x = 2 \)?

   \[
   f'(x) = \frac{x \cos x - \sin x}{x^2}
   \]

   \[
   f'(2) = \frac{2 \cos 2 - \sin 2}{4} \approx -0.354
   \]

   d. Find the equation of the tangent line to \( f \) at \( x = 2 \).

   P03. \( (a, \frac{\sin a}{a}) \) \hspace{1cm} S10P2 \( \frac{a \cos 2 - \sin 2}{4} \)

   \[
   (y - \frac{\sin a}{a}) = \frac{a \cos 2 - \sin 2}{4}(x - 2) \quad \text{OR: } \quad y = -0.354x + 1.3253
   \]

   e. Use local linearization to estimate \( \sin 2.1/2.1 \).

   Use tangent line from part (d): \( f(x) \approx -0.354x + 1.3253 \) (for \( x \) near \( a \))

   \[
   \sin 2.1 \approx f(2.1) = -0.41096
   \]

   OR: \( f(x) \approx f(a) + f'(a)(x - a) \)

   \[
   \frac{f(x) + f(2.1) + f'(2.1)(x - 2.1)}{2} \approx 0.41096
   \]
f. Do you expect your estimate to be above or below the actual value of \( \sin 2.1/2.1 \)? Explain.

Above, because the tangent lines is above the graph of \( f(x) \).
\( f(x) \) is concave down at \( x = 2 \).

\[ \lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \cos x = 1 \]

Using l'Hopital's Rule:

\( \lim_{x \to 0} \frac{\sin x}{x} = 1 \)

\( h. \) Define a function \( g(x) \) as

\[ g(x) = \begin{cases} f(x) & \text{if } x \neq 0 \\ 1 & \text{otherwise} \end{cases} \]

Is \( g(x) \) continuous at \( x = 0 \)? Explain.

Yes, \( g(x) \) is continuous at \( x = 0 \) because \( \lim_{x \to 0} g(x) = g(0) = 1 \).

\( i. \) Define a function \( G(x) \) as

\[ G(x) = \int_0^x f(t) \, dt \]

Find \( G(1) \).

\[ G(1) = \int_0^1 \frac{\sin t}{t} \, dt \approx 0.9461 \]

\( j. \) Using the same function \( g(x) \) defined above, what is \( G'(x) \)?

\[ G'(x) = \frac{d}{dx} \int_0^x \frac{\sin t}{t} \, dt = \frac{\sin x}{x} \]

(Second FTC)
2. **10 points** A closed box has a fixed surface area $A$ and a square base with side $x$.

a. Find a formula for the volume of the box, $V$, as a function of $x$.

$$
A = 2x^2 + 4xy \quad (A \text{ is fixed})
$$

So, $y = \frac{A - 2x^2}{4x}$

$$
V = x^2y \quad \Rightarrow \quad V(x) = x^2 \left( \frac{A - 2x^2}{4x} \right) = \frac{x}{4} (A - 2x^2)
$$

b. Find the maximum value of $V$.

$$
V(x) = \frac{1}{4} (A - 2x^2)
$$

$$
V'(x) = \frac{-1}{2} (A - 2x^2)
$$

Thus C.P. is max. because

$$
V''(x) = 0
$$

$$
V_{max} = \frac{1}{4} \left( A - \frac{A}{2} \right) = \frac{A}{8}
$$

3. **10 points** A car comes to a stop six seconds after the driver applies the brakes. While the brakes are on, the following velocities are recorded:

Time since brakes applied (sec) 0 2 4 6

Velocity (ft/sec) 80 40 12 0

a. Give a lower estimate for the distance the car traveled after the brakes were applied.

Since velocity is decreasing, lower estimate is RHS:

$$
40(2) + 12(2) + 0(2) = 104 \text{ feet}
$$

b. How often should the velocity be recorded if you wish to determine the actual distance the car traveled to within 2 feet?

$$
\text{We want } |v(t_1) - v(t_0)| \cdot \Delta t \leq 2
$$

So,

$$
|0 - 80| \cdot \Delta t \leq 2
$$

$$
\Delta t \leq \frac{2}{80} = \frac{1}{40}
$$

40 times per second, or every $\frac{1}{40}$ sec.
4. 10 points Find the solution of the following initial value problem:

\[ \frac{dy}{dx} = 6x^2 + 4x, \quad y(1) = 10 \]

\[ y = \frac{6x^3}{3} + \frac{4x^2}{2} + C = 2x^3 + 2x^2 + C \]

Since \( y(1) = 10 \), \( 2(1)^3 + 2(1)^2 + C = 10 \Rightarrow C = 6 \)

\[ y = 2x^3 + 2x^2 + 6 \]

5. 10 points Find \( \frac{dy}{dx} \) for each of the following:

a. \( y = 5x^2 + \pi^2 + e^{-x} \)

\( \frac{dy}{dx} = 10x + \pi \cdot \frac{d}{dx}(\ln \pi) - e^{-x} \)

b. \( y = \sqrt{e^2 - x^2} \)

\( \frac{dy}{dx} = \frac{1}{2\sqrt{e^2 - x^2}} = \frac{-x}{\sqrt{e^2 - x^2}} \)

c. \( y'' + yx^2 + x^2 = 3y^2 \)

\[ \frac{3y^2 \frac{dy}{dx} + y(2x) + x^2 \frac{dy}{dx} + 2x = 6y \frac{dy}{dx} \]

\[ \frac{dy}{dx} (3y^2 + x^2 - 6y) = -(2xy + 2x) \Rightarrow \frac{dy}{dx} = \frac{-(2xy + 2x)}{3y^2 + x^2 - 6y} \]

d. \( y = \tan(\arctan(kx)) \)

\( y = kx \), so \( \frac{dy}{dx} = k \)

\( \frac{dy}{dx} = \sec^2(\arctan(kx)) \cdot \frac{d}{dx} \arctan(kx) \cdot \frac{1}{1 + (kx)^2} \cdot k \)

Note: \( \frac{d}{dx} \arctan(kx) = \frac{k}{1 + (kx)^2} \). 

So, the 2 expressions are equivalent.
6. 10 points Evaluate the following. (Do all of these by hand, and show work. Do NOT use calculators except possibly to check your answers.)

a. $\int \frac{3}{x^2} \, dx = 3 \ln |x| + C$

b. $\int \frac{2-3x^2}{5x^2} \, dx = \left( \frac{2}{5x^2} \right) dx = \left( \frac{2}{5} x^{-2} \right) dx - \left( \frac{3}{5} dx \right)$
   \[= \frac{2}{5} x^{-1} - \frac{3}{5} x + C = -\frac{2}{5} x^{-\frac{3}{5}} x + C\]

c. $\int \left( \frac{1}{x^2} + \frac{1}{x} + \frac{1}{6x^2} \right) \, dx = \left( \frac{1}{x} \right) dx + \frac{x^{-1}}{-1} + x^{-\frac{1}{2}} + C$
   \[= \frac{1}{x} + \frac{x^{-\frac{1}{2}}}{x} + \frac{6x}{x^6} + C = 3 \cdot \frac{1}{x} + \frac{6x}{x^6} + C\]

d. $\int_0^{\pi} (\sin t + \cos t) \, dt$
   \[= \left[ \cos t + \sin t \right]^\pi_0 = \cos \pi + \sin \pi - (\cos 0 + \sin 0) = -(-1) - (0) = 2\]

e. $\frac{d}{dx} \int_0^x \cos \sqrt{t} \, dt = \cos \sqrt{x}$ (2nd Fundamental Theorem of Calculus)