1. Suppose $b$ has components $b_1, b_2, b_3$, $A$ is a $3 \times 4$ matrix and the augmented matrix corresponding to the equation $Ax = b$ is row equivalent to
\[
\begin{bmatrix}
1 & 4 & 6 & 8 & b_1 - 2b_3 \\
0 & 0 & 1 & 1 & -b_2 + 4b_3 \\
0 & 0 & 0 & 1 & 3b_1 + 2b_2 + b_3
\end{bmatrix}.
\]

1A. Suppose $k = 1$. What conditions (if any) must $b_1, b_2$ and $b_3$ satisfy in order for $b$ to be in $\text{Col}(A)$? Explain!
If $k = 1$, the system is consistent for any $b_1, b_2, b_3$. So there are NO restrictions (No conditions) on $b_1, b_2, b_3$.

1B. So, if $k = 1$, is $\text{Col}(A)$ all of $\mathbb{R}^3$? Explain!
Yes, $\text{Col}(A) = \mathbb{R}^3$ since for any $b \in \mathbb{R}^3$, $A\vec{x} = b$ has a solution, i.e., $b$ can be expressed as a L.C. of the columns of $A$.

1C. Suppose $k = 1$. Find vectors that span the nullspace of $A$. Hint: Think about the way we write the solutions of the homogeneous equation $Ax = 0$ in "parametric form".
\[
\begin{bmatrix}
1 & 4 & 6 & 8 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 & 0 & -4 \\
0 & 1 & 0 & -2 \\
0 & 0 & 1 & 0
\end{bmatrix} \Rightarrow A\vec{x} = 0 \text{ has solutions } \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4x_4 \\ 2x_4 \\ x_4 \\ 0 \end{bmatrix} = x_4 \begin{bmatrix} 4 \\ 2 \\ 1 \\ 0 \end{bmatrix}
\]
where $x_4$ is free. All vectors in $\text{null}(A)$ are multiples of the (single) vector $[4, 2, 1, 0]^T$; this vector spans the nullspace.

1D. Suppose $k = 0$. What conditions (if any) must $b_1, b_2$ and $b_3$ satisfy in order for $b$ to be in $\text{Col}(A)$? Explain!
Now $A\vec{x} = b$ has a solution $\iff 3b_1 + 2b_2 + b_3 = 0$ (otherwise the system is inconsistent)

1E. So, if $k = 0$, is $\text{Col}(A)$ all of $\mathbb{R}^3$? Explain!
No. Only vectors $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ satisfying $\text{THIS}$ are in the column space now. \{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \in \mathbb{R}^3 \text{ but } \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \notin \text{ Col}(A) \}

2. Let $F$ be the vector space of all continuous functions $f : \mathbb{R} \rightarrow \mathbb{R}$, as discussed in class.

2A. Is the set $H = \{ f \in F \mid \text{the graph of } f \text{ passes through the point } (0, 3) \}$ closed under vector addition? Prove it or give a counterexample.

\textbf{It is NOT closed.} For (counter)example, let $f(x) = x + 3$ and $g(x) = x^2 + 3$.
Since $f(0) = 3, f \in H$, and since $g(0) = 3, g \in H$, but $(f + g)(0) = f(0) + g(0) = 6$, i.e., the graph of $f + g$ passes through $(0, 6)$ instead of $(0, 3)$, so $(f + g) \notin H$.
We've found two specific members of $H$ whose sum is $\text{NOT}$ in $H$.

2B. Is the set $G = \{ f \in F \mid \text{the graph of } f \text{ passes through the point } (3, 0) \}$ closed under vector addition? Prove it or give a counterexample.

\textbf{It IS CLOSED}.
Let $f$ and $g$ be any arbitrary members of $G$; we need to show $f + g \in G$.
So $f(3) = 0$ and $g(3) = 0$. Now, $(f + g)(3) = f(3) + g(3) = 0 + 0 = 0$.
So $f + g$ also passes through $(3, 0)$ and thus $f + g \in G$.

2C. Which (if either) of $H$ or $G$ is a subspace of $F$? \(H \text{ is NOT since it's not closed under addition, and doesn't even contain the 0 vector!!!)\)
\(\text{Note: } 0 \in G \text{ since the constant function } 0 \text{ passes through } (3, 0); \text{ it is closed under addition AND it's easy to show it's closed under scalar multi. too).}