1. Start by using the ratio test:

\[ a_n = \frac{(x-3)^n}{n(n+1)}, \quad \text{so} \quad a_{n+1} = \frac{(x-3)^{n+1}}{(n+1)(n+2)} \]

and therefore

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(x-3)^{n+1}}{(n+1)(n+2)} \cdot \frac{n(n+1)}{(x-3)^n} \right| \\
= \lim_{n \to \infty} \left| \frac{(x-3)}{n+2} \right| \\
= |x-3| \lim_{n \to \infty} \frac{1}{1 + \frac{2}{n}} \\
= |x-3| \cdot \frac{1}{1+0} = |x-3|.
\]

So the series converges if \(|x-3| < 1\), it diverges if \(|x-3| > 1\), and we don’t know yet what it does if \(|x-3| = 1\). If \(|x-3| = 1\) then either \(x-3 = 1\) or \(x-3 = -1\); in other words, either \(x = 4\) or \(x = 2\). We know so far that the series converges if \(2 < x < 4\) and that the only other places where it might converge are \(x = 2\) and \(x = 4\). Plugging \(x = 4\) into the series we get

\[
\sum_{n=1}^{\infty} \frac{(4-3)^n}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}.
\]

This converges, either because

\[
\sum_{n=1}^{\infty} \frac{1}{n(n+1)} < \sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{which is finite (it equals } \frac{\pi^2}{6} \text{)}
\]

or because

\[
\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{n+1-n}{n(n+1)} = \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right) = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \frac{1}{5} - \frac{1}{6} + \cdots = 1.
\]

When \(x = 2\) the series becomes

\[
\sum_{n=1}^{\infty} \frac{(2-3)^n}{n(n+1)} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)}.
\]

This converges by the alternating series test because the non-alternating part, namely \(\frac{1}{n(n+1)}\), decreases to zero. We could also say it converges absolutely (which implies that it converges), since if we put absolute values on the terms we would get \(\sum_{n=1}^{\infty} \frac{1}{n(n+1)}\), which we already know converges. So this series converges at both endpoints: the interval of convergence is \(2 \leq x \leq 4\).

I didn’t ask this, but we can find what function the series converges to by using the \(1 = n + 1 - n\) trick:

\[
\sum_{n=1}^{\infty} \frac{(x-3)^n}{n(n+1)} = \sum_{n=1}^{\infty} \frac{n+1-n}{n(n+1)} (x-3)^n = \sum_{n=1}^{\infty} \frac{(x-3)^n}{n} - \sum_{n=1}^{\infty} \frac{(x-3)^n}{n+1}.
\]

We know that

\[
(L) \quad -\ln(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n} \quad \text{for } -1 < z < 1
\]
which will give us the first series above. To get the second one, rewrite (L) as
\[-\frac{\ln(1 - z)}{z} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{n} = \sum_{n=0}^{\infty} \frac{z^n}{n+1}.\]

Since the first term of this series is 1, if we subtract 1 from both sides we get
\[\text{(LL)} \quad -\frac{\ln(1 - z)}{z} - 1 = \sum_{n=1}^{\infty} \frac{z^n}{n+1}.\]

Taking \(z = x - 3\) in (L) and (LL) we get
\[
\sum_{n=1}^{\infty} \frac{(x - 3)^n}{n} - \sum_{n=1}^{\infty} \frac{(x - 3)^n}{n + 1} = -\ln(1 - (x - 3)) + \frac{\ln(1 - (x - 3))}{x - 3} + 1
= 1 + \frac{\ln(4 - x)}{x - 3} = 1 + \frac{x - 4}{x - 3} \ln(4 - x).
\]

This holds if \(2 \leq x \leq 4\), although one needs to take a limit if \(x = 4\).

2. We have to plug into the formula
\[f(a) + f'(a)(x - a) + f''(a) \frac{(x - a)^2}{2!} + f'''(a) \frac{(x - a)^3}{3!}\]
with \(a = 7\) and \(f(x) = \frac{1}{8 - x}\). We can plug the 7 in right away: we need to work out
\[f(7) + f'(7)(x - 7) + f''(7) \frac{(x - 7)^2}{2!} + f'''(7) \frac{(x - 7)^3}{3!}\]

In particular we need the first three derivatives of \(f(x) = (8 - x)^{-1}\). Using the chain rule, we have
\[
\begin{align*}
f'(x) &= (-1)(8 - x)^{-2}(-1) = (8 - x)^{-2} = \frac{1}{(8 - x)^2} \\
f''(x) &= (-2)(8 - x)^{-3}(-1) = 2(8 - x)^{-3} = \frac{2}{(8 - x)^3} \\
f'''(x) &= 2(-3)(8 - x)^{-4}(-1) = 6(8 - x)^{-4} = \frac{6}{(8 - x)^4}
\end{align*}
\]

Plugging in \(x = 7\) we get \(f(7) = 1, f'(7) = 1, f''(7) = 2\) and \(f'''(7) = 6\). Putting these numbers in formula (T) we get the answer
\[
1 + (x - 7) + 2 \frac{(x - 7)^2}{2!} + 6 \frac{(x - 7)^3}{3!} = 1 + (x - 7) + (x - 7)^2 + (x - 7)^3.
\]

We can get the full Taylor series by realizing that
\[f^{(n)}(x) = \frac{n!}{(8 - x)^{n+1}}\]
and therefore \[f^{(n)}(7) = \frac{n!}{(8 - 7)^{n+1}} = n!\]

Therefore the general formula
\[
\sum_{n=0}^{\infty} f^{(n)}(7) \frac{(x - 7)^n}{n!}
\]
becomes \[
\sum_{n=0}^{\infty} n! \frac{(x - 7)^n}{n!} = \sum_{n=0}^{\infty} (x - 7)^n.
\]

Another way to get this is to recall that
\[
\frac{1}{1 - r} = \sum_{n=0}^{\infty} r^n \quad \text{if} \quad -1 < r < 1.
\]

If we replace \(r\) by \(x - 7\) then this becomes
\[
\frac{1}{8 - x} = \frac{1}{1 - (x - 7)} = \sum_{n=0}^{\infty} (x - 7)^n \quad \text{if} \quad -1 < x - 7 < 1, \quad \text{which means} \quad 6 < x < 8.
\]