1. Let $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -5 & 3 & 1 \end{bmatrix}$, $U = \begin{bmatrix} 2 & 3 & 0 & 4 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, and $A = \begin{bmatrix} 2 & 3 & 0 & 4 \\ 4 & 9 & 1 & 8 \\ -10 & 3 & -19 \end{bmatrix}$. Suppose $A = LU$.

1A. What is the value of $t$? (What expression did you evaluate to find it?)

$$t = \text{"(row 3 of L) \times (col 2 of U)"} = -5 \cdot 3 + 3 \cdot 3 + 1 \cdot 0 = -15 + 9 = -6$$

1B. Use the method of $LU$ decomposition as discussed in class to solve $Ax = b$, where $b = \begin{bmatrix} 4 \\ 2 \\ -37 \end{bmatrix}$.

Show all your steps. Write the solution as $p + v_h$, where $p$ is a particular solution of $Ax = b$ and $v_h$ gives the solution(s) of the corresponding homogeneous equation.

$$A \hat{x} = b \Rightarrow$$

so $\hat{y} = \begin{bmatrix} 4 \\ -6 \end{bmatrix}$ and $\hat{y} = U \hat{x}$; solve $\begin{bmatrix} 2 & 3 & 0 & 4 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \end{bmatrix}$

now $x_4 = 1$

$3x_2 + x_3 = -6 \Rightarrow x_2 = \frac{-6 - x_3}{3} = \frac{-2 - x_3}{1}$

$5x_1 + 3x_2 + x_3 = -37 \Rightarrow x_3 = \frac{-20 - 18x_1 - 37}{2}$

$3x_1 + x_2 = -6 \Rightarrow x_2 = \frac{-6 - 2}{3}$

1C. The augmented matrix $[A|b]$ is row equivalent to $\begin{bmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Use this to again write the solution of $Ax = b$ as $p + v_h$. How does this solution compare with the one from 1B (is the particular solution the same?)

here $\hat{x} = \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \end{bmatrix}$ where $x_2$ is free, that is, $\hat{x} = \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1/2 \\ 0 \\ 1/3 \end{bmatrix}$; same as 1B's particular solution."

1D. What one elementary row operation changes $A$ into $\begin{bmatrix} 2 & 3 & 0 & 4 \\ 4 & 9 & 1 & 8 \\ 0 & 3 & 1 \end{bmatrix}$? (Describe it: "Row 3 is replaced by ...")

$R_3$ is replaced by $R_3 + 5R_1$.

1E. What is the value of $t$ after this operation? in $A_t$, $t$ is $-6$ and $x = -6 + 5 \cdot 3 = -6 + 15 = 9$

1F. What elementary matrix $E$ corresponds to, or, carries out this operation, if you compute $EA$?

Do to $I_3$ what you want to do to $A$, that is, replace its third row by $R_3 + 5R_1$. Since $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, this yields $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix}$.
2. Let \( B = \begin{bmatrix} 9 & 7 & 4 & -9 \\ 0 & 0 & 5 & 8 \\ 0 & 0 & 6 & 10 \\ 11 & 0 & 21 & 90 \end{bmatrix} \).

2a. Find the determinant of \( B \). Expand using any row or column you like, but look at all those 0's in the second column!

\[
\text{So } \det(B) = -7 \begin{vmatrix} 0 & 5 & 8 \\ 11 & 21 & 90 \end{vmatrix} + 0 \begin{vmatrix} ? & ? \\ 11 & 21 \end{vmatrix} + 0 \begin{vmatrix} ? & ? \\ 11 & 21 \end{vmatrix} 
\]

\[
= -7 \cdot 11 \begin{vmatrix} 5 & 8 \\ 6 & 10 \end{vmatrix} = -7 \cdot 11 \cdot (50 - 48) = -7 \cdot 11 \cdot 2 = -154
\]

2b. Find \( \det(B^T) \).

\[
\det(B^T) = \det(B) = -154
\]

2c. Find \( \det(B^{-1}) \).

\[
\det(B^{-1}) = \frac{1}{\det(B)} = \frac{-1}{154}
\]

(provided \( \det(B) \neq 0 \))