1. Suppose \[ \begin{bmatrix} -8 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 4 \\ 0 \\ 0 \\ -3 \\ 1 \end{bmatrix}, \] where \( x_3 \) and \( x_5 \) are free, is the solution \( \mathbf{v}_h \) of the homogeneous matrix equation \( B \mathbf{x} = \mathbf{0} \) for some matrix \( B \). Also, let \( \mathbf{v}_1, \mathbf{v}_2, \ldots \), be the column vectors of \( B \).

1A. You can not tell from the above info how many rows \( B \) has. But how many columns must \( B \) have, and how do you know?

The six rows in the vectors above tell us the variables are \( x_1, x_2, \ldots, x_6 \) one for each column of \( B \). If \( B \) has 6 columns.

1B. Is the set \( \{ \mathbf{v}_1, \mathbf{v}_2, \ldots \} \) of column vectors of \( B \) linearly independent? Explain in terms of the definition of LI (that is, consider the solutions of \( x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 + \cdots = \mathbf{0} \)).

No. In order to be a LI set, the only solution to \( x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 + \cdots + x_6 \mathbf{v}_6 = \mathbf{0} \) must be the trivial solution, that is, \( x_1 = \cdots = x_6 = 0 \). But since \( x_2 \) and \( x_5 \) are free, this \( \mathbf{v}_h \) has non-trivial solutions.

1C. Use the equation \( x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 + x_3 \mathbf{v}_3 = \mathbf{0} \) to express \( \mathbf{v}_1 \) as a specific linear combination of the other column vectors, or explain why this is impossible.

We need some values of \( x_1, \ldots, x_6 \) which satisfy this equation and where \( x_i \neq 0 \). There are many ways to do this. For example, let \( x_2 = x_4 = 1 \). Then we also have \( x_1 = -81 + 41 = -40, x_2 = 0, x_3 = 0 \), and we can obtain \( \mathbf{v}_1 = -\mathbf{40} + \mathbf{0} + \mathbf{1} = \mathbf{1} \) and \( \mathbf{v}_2 = \mathbf{1} + \mathbf{0} = \mathbf{1} \).

1D. Express \( \mathbf{v}_2 \) as a linear combination of the other column vectors, or explain why this is impossible.

\[ \text{Impossible: if } \mathbf{v}_2 = \alpha \mathbf{v}_1 + \beta \mathbf{v}_2 + \gamma \mathbf{v}_3 + \delta \mathbf{v}_4 \text{ then } \delta = \alpha \mathbf{v}_2 + \beta \mathbf{v}_3 + \gamma \mathbf{v}_4 = \mathbf{0} \text{, a solution of } B \mathbf{x} = \mathbf{0} \text{ in which } x_2 \neq 0 \] for \( \alpha, \beta, \gamma, \delta \) real numbers.

1E. Let \( \mathbf{b} = 7 \mathbf{v}_1 + 6 \mathbf{v}_2 - 12 \mathbf{v}_4 \) in the following two questions:

1E(i). Express \( \mathbf{b} \) as a linear combination of the column vectors of \( B \) without using \( \mathbf{v}_4 \) (by replacing \( \mathbf{v}_4 \) so we have a LC of the other column vectors).

\[ \text{We first need to express } -12 \mathbf{v}_4 \text{ in terms of the other columns.} \]

Now, if \( x_5 = 1 \) and \( x_3 = 0 \), we get \( x_1 = -8 x_3 + 4 x_5 = -80 + 41 = 49 \), \( x_2 = 0, x_3 = 0 \), \( x_4 = -3 x_5 = -3, x_5 = 1, x_6 = 0 \).

\[ \text{so } 49, -3, -49, 0 \text{ is a solution of } B \mathbf{x} = \mathbf{0} \text{ in which } x_2 \neq 0 \]

1E(ii). Can you express \( \mathbf{b} \) as a linear combination of the column vectors of \( B \) without using \( \mathbf{v}_2 \)? Explain your answer. No, since \( \mathbf{v}_2 \) is not a LC of the other columns.

**Better:** If \( B \) is a LC of other columns, then \( \mathbf{b} = B \mathbf{v}_2 \) (i.e., all columns) \(-12 \mathbf{v}_4 \); then the last equation gives

1 Challenge Bonus Question: Let \( m \) be the number of rows of \( B \). Suppose \( B \mathbf{x} = \mathbf{c} \) does not have a solution for every \( \mathbf{c} \) in \( \mathbb{R}^m \). What is the RREF of \( B \), where \( m \) is as small as possible?

We have \( \begin{cases} x_1 + 8 x_3 - 4 x_5 = 0 \\ x_2 + 3 x_3 = 0 \\ x_4 + 3 x_5 = 0 \\ x_6 = 0 \end{cases} \) from \( \mathbf{v}_1 \); in RREF(\( B \)) is

\[ \begin{bmatrix} 1 & 0 & 0 & -4 & 0 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \]

Then we add a row of \( 0 \)'s to ensure \( B \mathbf{x} = \mathbf{c} \) might be inconsistent.