1. (9 pts.) Consider the differential equation

\[ y' = x + y. \]

(a) (2 pts.) Accurately draw the slope field at the nine points where \( x \) and \( y \) are \(-1, 0, \) and \( 1 \). (It’s easiest to do this without using your calculator, but you can use your calculator program to check your work.)

(b) (5 pts.) The solution through the point \((0, 1)\) is not obvious. However, you can find it numerically by Euler’s method. Using a step size of \( \Delta x = 0.1 \), find estimates of points on the solution curve up to where \( x = 0.3 \).

(c) (2 pts.) From looking at the slope field (using your calculator program), do you think the \( y \)-values you found in part (b) are too large or too small? Explain why.

2. (8 pts.) The region in the first quadrant bounded by the \( x \)-axis, the \( y \)-axis, and the graph of \( y = 25 - x^2 \) is rotated around the line \( x = -3 \). Give, but DO NOT EVALUATE, an integral expressing the volume of the resulting solid.
3. (9 pts.) A developer intends to construct a pond in order to make the homes he is building more attractive. By diverting a stream, he can supply the pond with about 2000 gallons of water per day. On the other hand, a study of soil conditions has shown he should expect a continuous loss of water through seepage into the ground at a rate of 4% of the amount in the pond per day.

(a) (2 pts.) Write a differential equation describing how \( A(t) \), the amount of water in the pond at time \( t \), will behave.

(b) (2 pts.) What value of \( A \) is an equilibrium for this differential equation? Will the equilibrium be stable or unstable? Explain by using your calculator to produce a slope field, and sketching the relevant part of that field here.

(c) (5 pts.) Assuming the pond initially has no water in it, find the solution to your differential equation.

4. (6 pts. – 2 pts. each)

(a) Give a Taylor series for \( e^x \) about the point \( a = 0 \). Write out enough terms so that the pattern is clear.

(b) Give a Taylor series for \( f(x) = xe^{-3x} \) about \( a = 0 \). Write out enough terms so that the pattern is clear.

(c) Use the Taylor polynomial of degree 3 about the point \( a = 0 \) to estimate \( f(-0.1) \).
5. (16 pts. – 4 pts. each) Evaluate the following integrals. Do NOT use any numerical integration techniques, except perhaps to check your answers. If you refer to the table of integrals, you must cite any formulas you use.

(a) \[ \int \frac{1}{x - x^2} \, dx \]

(b) \[ \int \frac{\cos \sqrt{x}}{\sqrt{x}} \, dx \]

(c) \[ \int_{0}^{\pi} x \sin x \, dx \]

(d) \[ \int_{2}^{5} \frac{x}{\sqrt{x^2 - 1}} \, dx \]
6. (7 pts.) Suppose you win a million dollars ($1,000,000) in a lottery, and are told your winnings will be
paid out continuously over the next 20 years at a rate of $50,000 per year.

(a) (1 pt.) Should the present value of this income stream be less than, equal to, or more than
$1,000,000?

(b) (6 pts.) Compute the present value of the income stream, assuming the prevailing interest rate is
5%, compounded continuously.

7. (10 pts. – 5 pts. each) Solve the following differential equations. Be sure to use the initial conditions
to determine the values of any constants you introduce.

(a) \( \frac{dy}{dt} = kt - yt \) with \( y(0) = 10 \).

(b) \( \frac{dz}{dt} = te^z \) with \( z(2) = 0 \).
8. (8 pts. – 2 pts. each) Complete the following.

(a) If a function is decreasing, but concave up, then when a midpoint sum is used to approximate its integral the approximation will be too _________ because . . .

(b) Taylor polynomials are designed to have certain features in common with the function they are approximating. If \( p_3(x) \) is the third degree Taylor polynomial approximating \( f(x) \) for \( x \) near \( a \), then the graphs of \( p_3(x) \) and \( f(x) \) will have the following features in common: . . .

(c) The series \( 1 + x + x^2 + x^3 + x^4 + \ldots \) converges for all \( x \) such that _______. For such an \( x \) the series converges to _______.

(d) The Alternating Series Test states that . . .

9. (8 pts.)

When a volcano erupts, it spreads ash over the surrounding land. Suppose a number of measurements are made and the density of ash deposited at a distance \( r \) meters from the center of the crater is found to be described well by the function

\[
\rho(r) = \frac{2000}{1 + r^2} \text{kg/m}^2.
\]

Give and evaluate an integral for the total amount of ash deposited in the circular region within 1000 km of the volcano.
10. (15 pts. – 3 pts. each) Do the following converge to finite values, or not? Justify your answers.

(a) \( \int_1^\infty \frac{x^2}{1 + 2 \ln x + 3x^5} \, dx \)

(b) \( \int_2^3 \frac{1}{6 - 2t} \, dt \)

(c) \( \sum_{n=1}^\infty \frac{1}{\sqrt{n}} \)

(d) \( \sum_{n=1}^\infty \frac{(-1)^{n+1}}{\sqrt{n}} \)

(e) \( \sum_{n=1}^\infty \frac{1}{5^n + n^9} \)

11. (4 pts.) What is the radius of convergence of the series \( \sum_{n=1}^\infty \frac{2n}{5^n} x^n \)?