Show all your work. On each problem you must write down enough details of what you have done that your method is understandable.

1. (20 pts. – 4 pts. each) State whether the following converge or diverge. Justify your answer by using a convergence test. You may also use any memorized facts you may know about either geometric series or $p$-series.

(a) \[ \sum_{n=0}^{\infty} \frac{n}{n+2} \]

(b) \[ \frac{5}{2} - \frac{5}{3} + \frac{5}{4} - \frac{5}{5} + \frac{5}{6} \ldots \]

(c) \[ \sum_{n=1}^{\infty} \frac{n^3}{3^n} \]

(d) \[ \sum_{n=2}^{\infty} \frac{1}{n \ln(n)} \]

(e) \[ \sum_{n=2}^{\infty} \frac{n + 2}{2n^3 + 1} \]
2. (10 pts. total) A bouncing ball travels a total of 2 meters in its first bounce (up and down). Each additional bounce covers exactly $\frac{3}{5}$ of the distance of the bounce immediately preceding it. Let $B_n$ denote the total distance traveled in the first $n$ bounces, so $B_1 = 2$

(a) (1 pt.) Find $B_2$, $B_3$, and $B_4$. You may leave your answers as sums.

(b) (4 pts.) Give a general formula for $B_n$, in closed form.

(c) (1 pt.) Give a decimal value for $B_{12}$, accurate to at least six digits after the decimal point.

(d) (4 pts.) What is the total distance the ball will have traveled if it is allowed to bounce ‘forever’?

3. (16 pts. – 8 pts. each) Use whatever convergence tests you like to justify your answers. (However, you may not simply quote a memorized result as justification.)

(a) Indicate whether $\sum_{n=1}^{\infty} \frac{1}{n}$ converges or diverges, and justify your answer.

(b) Indicate for which $p > 1$ the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges, and for which $p > 1$ the series converges. Then justify your answer.
4. (13 pts. total)

(a) (2 pts.) Give the general formula for the Taylor series about $a$ for a function $f(x)$.

(b) (7 pts.) Use your answer in (a) to give the first few terms of its Taylor series about $a = 1$ for $f(x) = \sqrt{x}$. Write the series in a form where the terms have a clear pattern. (If you are not sure you have written enough terms, then you should probably write one more.)

(c) (4 pts.) Use the third degree Taylor polynomial approximating $\sqrt{x}$ near $a = 1$ to find an approximate value for $\sqrt{0.9}$. Your final answer should be a numerical value, showing at least six digits.

5. (10 pts. – 5 pts. each) Give Taylor series for the following, using any method you like.

(a) $f(x) = x \sin(x^2)$

(b) $g(x) = \frac{2}{1 + 3x}$
6. (16 pts. total) Consider the function \( f(x) \) defined by the series
\[
f(x) = (x - 1) + \frac{(x - 1)^2}{(2)(2)} + \frac{(x - 1)^3}{(2^2)(3)} + \frac{(x - 1)^4}{(2^3)(4)} + \frac{(x - 1)^5}{(2^4)(5)} + \frac{(x - 1)^6}{(2^5)(6)} + \ldots
\]

(a) (5 pts.) Determine the radius of convergence for the series.

(b) (3 pts.) What does your answer in part (a) mean? That is, for what \( x \) do you know the series converges? for what \( x \) do you know the series diverges?

(c) (3 pts.) Give a series for \( f'(x) \).

(d) (5 pts.) The series for \( f'(x) \) should be recognizable to you. Express it in closed form.

7. (15 pts. – 5 pts. each) The following statements may be true or false, or somewhere in between. Briefly discuss each statement, either correcting it, or indicating why it is correct. It may be helpful to give examples. (Points are awarded for the overall quality and completeness of your answer.)

(a) If the terms of a series get smaller, then the series must converge.

(b) Any alternating series converges.

(c) By looking at a Taylor series about \( a = 0 \) for a function, we can sometimes tell something about the symmetry of the function’s graph.