1. If you use the ratio test you’ll get \( \frac{2}{3} \), so the series converges, but we can say more. This is a geometric series with \( r = \frac{2}{3} \) and \( a = 1 \); or, to put it another way, it is an instance of

\[
(G) \quad \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{if } -1 < x < 1
\]

with \( x = \frac{2}{3} \). Therefore

\[
\sum_{n=0}^{\infty} \frac{2^n}{3^n} = \frac{1}{1 - \frac{2}{3}} = \frac{3}{1} = 3.
\]

Although we know that some series (\( p \)-series with \( 0 < p \leq 1 \), for example) don’t converge even though the terms go to zero, a geometric series always converges when the terms go to zero.

2. Use the ratio test with \( a_n = \frac{(4n)!}{(2n)! \, (n!)^2 \, (48)^n} \), so that

\[
a_{n+1} = \frac{(4(n+1))!}{(2(n+1))! \, [(n+1)!]^2 \, (48)^{n+1}} = \frac{(4n+4)!}{(2n+2)! \, [(n+1)!]^2 \, (48)^{n+1}}.
\]

Since \( a_n \) and \( a_{n+1} \) are always positive, we won’t need the absolute values in the statement of the ratio test. We have

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{(4n+4)!}{(2n+2)! \, (n+1)!} \frac{(2n)! \, (n!)^2 \, (48)^n}{(4n)!} = \lim_{n \to \infty} \frac{(4n+4)!}{(4n)!} \frac{(2n)! \, (n!)^2 \, (48)^n}{(2n+2)! \, (n+1)! \, (48)^{n+1}}.
\]

Here we get the usual factorial cancellations:

\[
\frac{(m+k)!}{m!} = \frac{1 \cdot 2 \cdots (m)(m+1) \cdots (m+k)}{1 \cdot 2 \cdots (m)} = (m+1)(m+2) \cdots (m+k),
\]

so

\[
\frac{(4n+4)!}{(4n)!} = (4n+1)(4n+2)(4n+3)(4n+4) \quad \text{and} \quad \frac{(2n)!}{(2n+2)!} = \frac{1}{(2n+1)(2n+2)} \quad \text{and} \quad \frac{n!}{(n+1)!} = \frac{1}{n+1}
\]

and therefore

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{(4n+1)(4n+2)(4n+3)(4n+4)}{(2n+1)(2n+2)(n+1)(48)} = \lim_{n \to \infty} \frac{4n+1 \, 4n+2 \, 4n+3 \, 4n+4}{n+1 \, 2n+1 \, n+1 \, 2n+2 \, 48} = \lim_{n \to \infty} \frac{1 \, 4n+1 \, 4n+3}{12 \, n+1 \, n+1}
\]

because two of the numerator factors are twice a corresponding denominator factor. If \( b \) is any constant we have

\[
\lim_{n \to \infty} \frac{4n+b}{n+1} = \lim_{n \to \infty} \frac{4}{1} = 4,
\]

so

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{4n+1 \, 4n+3 \, 1 \, 12 - 4 \cdot 4}{12} = 3 > 1.
\]
Therefore the series \( \sum_{n=0}^{\infty} \frac{(4n)!}{(2n)! (n!)^2 (48)^n} \) diverges. The ratio test calculation shows that each term is roughly \( \frac{4}{3} \) times the preceding term, so the terms don’t get small. They do decrease at the start:

\[
\sum_{n=0}^{\infty} \frac{(4n)!}{(2n)! (n!)^2 (48)^n} = 1 + \frac{1}{4} \cdot \frac{35}{192} + \frac{385}{2304} + \frac{25025}{147456} + \ldots
\]

and the smallest term in the series is \( \frac{385}{2304} \). Theoretically the \( n \)th term test would also work on this, but the terms are so complicated that the ratio test is easier.

3. All three of these series are alternating, but the alternating series test only works on one of them.

3(a) This is the classic alternating series setup: the nonalternating part is \( \frac{1}{\sqrt{24n+1}} \), which decreases to zero as \( n \to \infty \) since \( 24n + 1 \) increases to \( \infty \) as \( n \to \infty \). Therefore the series converges. It’s kind of an interesting series, actually. The first several terms are

\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{24n+1}} = 1 - \frac{1}{5} + \frac{1}{7} - \frac{1}{\sqrt{73}} + \frac{1}{\sqrt{97}} - \frac{1}{11} + \frac{1}{\sqrt{145}} - \frac{1}{13} + \ldots
\]

Here’s an extra credit problem you can try: explain why the reciprocal of every positive odd number, except the multiples of 3, is in this series. (We’ve seen the reciprocals of 1, 5, 7, 11, 13 so far.) The degree of difficulty here is about that of an integral or sum of the week, but the mathematics involved is algebra, rather than calculus.

3(b) This alternates, and the nonalternating part is

\[
\frac{n}{41n-1} = \frac{1}{41 - \frac{1}{n}}.
\]

Since \(-\frac{1}{n}\) increases to zero as \( n \) increases, \( 41 - \frac{1}{n} \) increases to 41, and therefore

\[
\frac{n}{41n-1} = \frac{1}{41 - \frac{1}{n}}
\]

decreases as \( n \) increases. But it doesn’t decrease to zero:

\[
\lim_{n \to \infty} \frac{n}{41n-1} = \lim_{n \to \infty} \frac{1}{41 - \frac{1}{n}} = \frac{1}{41 - 0} = \frac{1}{41}.
\]

(Or L’Hopital’s rule could be used.) Therefore the series diverges by the \( n \)th term test.

3(c) This is similar to an example we did in class. The series alternates, and the terms go to zero, but the even terms go there faster than the odd terms so there is not a consistent decrease. The positive terms diverge:

\[
1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{15} + \ldots > \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \frac{1}{12} + \frac{1}{14} + \frac{1}{16} + \ldots = \frac{1}{2} \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \ldots\right) = \infty,
\]

since it is half the \( p \)-series with \( p = 1 \). The negative terms converge:

\[
-\frac{1}{2^2} - \frac{1}{4^2} - \frac{1}{6^2} - \frac{1}{8^2} - \frac{1}{10^2} - \frac{1}{12^2} - \frac{1}{14^2} - \frac{1}{16^2} - \ldots = -\frac{1}{2^2} \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{8^2} + \ldots\right) = \frac{1}{2} \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{8^2} + \ldots\right).
\]
Brian Dupee’s argument was essentially limit comparison: the series should behave like

\[ \frac{1}{n^{3/2}}. \]

When conceived of this way, this is not an alternating series any more, but rather a series of positive terms.

There are several possible answers to the extra credit. Madeline O’Donnell’s answer was

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \frac{1}{n^{3/2}} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(\sqrt{n})^{3/2}} = 1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} + \frac{1}{5} - \frac{1}{6^2} + \frac{1}{7} - \frac{1}{8^2} + \cdots \]

and this is also the first thing I thought of. My other answer was

\[ \sum_{n=1}^{\infty} \frac{(-1)^n \sin \frac{n\pi}{2}}{n} = 1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} + \frac{1}{5} - \frac{1}{6^2} + \frac{1}{7} - \frac{1}{8^2} + \cdots \]

and Oleg Alekseev had something similar:

\[ \sum_{n=1}^{\infty} \frac{2}{n [\sqrt{n(1-(-1)^n)} + 1 - (-1)^n]} = 1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} + \frac{1}{5} - \frac{1}{6^2} + \frac{1}{7} - \frac{1}{8^2} + \cdots \]

Vincent Roux’s answer was

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \frac{1}{n^{3/2}} = 1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} + \frac{1}{5} - \frac{1}{6^2} + \frac{1}{7} - \frac{1}{8^2} + \cdots \]

and Nik Dettman’s answer was

\[ \sum_{n=1}^{\infty} \frac{2}{n(n+1)(1+(-1)^n) + 1 - (-1)^n} = 1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} + \frac{1}{5} - \frac{1}{6^2} + \frac{1}{7} - \frac{1}{8^2} + \cdots \]

A number of you were regrouping the series, which is a good idea. Since

\[ 1 + \frac{1}{3} + \frac{1}{5} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2n-1} \quad \text{and} \quad - \frac{1}{2^2} - \frac{1}{4^2} - \frac{1}{6^2} - \cdots = - \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = - \sum_{n=1}^{\infty} \frac{1}{4n^2} \]

we have

\[ \left( 1 - \frac{1}{2^2} \right) + \left( \frac{1}{3} - \frac{1}{4^2} \right) + \left( \frac{1}{5} - \frac{1}{6^2} \right) + \cdots = \sum_{n=1}^{\infty} \left( \frac{1}{2n-1} - \frac{1}{4n^2} \right) = \sum_{n=1}^{\infty} \frac{4n^2 - 2n + 1}{8n^3 - 4n^2}. \]

When conceived of this way, this is not an alternating series any more, but rather a series of positive terms. So we can’t use the alternating series test any more, but now we can use comparison tests (or even the integral test), which we couldn’t before. By ordinary comparison

\[ \left( 1 - \frac{1}{2^2} \right) + \left( \frac{1}{3} - \frac{1}{4^2} \right) + \left( \frac{1}{5} - \frac{1}{6^2} \right) + \cdots = \sum_{n=1}^{\infty} \frac{4n^2 - 2n + 1}{8n^3 - 4n^2} > \sum_{n=1}^{\infty} \frac{4n^2 - 2n}{8n^3 - 4n^2} = \sum_{n=1}^{\infty} \frac{1}{2n} = \infty. \]

Brian Dupee’s argument was essentially limit comparison: the series should behave like \( \frac{1}{2n^2} \). The calculation

\[ \lim_{n \to \infty} \frac{\frac{2n-1}{2n-1} - \frac{2n^2}{2n-1}}{\frac{2n}{2n-1}} = \lim_{n \to \infty} \frac{1}{2n-1} - \lim_{n \to \infty} \frac{1}{2n-1} = \lim_{n \to \infty} \left( 1 - \frac{2n-1}{4n^2} \right) = 1 - \lim_{n \to \infty} \frac{2}{8n} = 1 - 0 = 1 \]
confirms this. Since \( \sum_{n=1}^{\infty} \frac{1}{n^{p-1}} \) diverges (by (H)), so does
\[
\left(1 - \frac{1}{2^2}\right) + \left( \frac{1}{3} - \frac{1}{4^2}\right) + \left( \frac{1}{5} - \frac{1}{6^2}\right) + \cdots = \sum_{n=1}^{\infty} \left( \frac{1}{2n-1} - \frac{1}{4n^2}\right).
\]
The integral test also works well on the regrouped series:
\[
\int_{1}^{\infty} \left( \frac{1}{2x-1} - \frac{1}{4x^2}\right) \, dx = \frac{1}{2} \ln(2x-1) + \frac{1}{4x}\bigg|_{1}^{\infty} = \infty + 0 - \left(0 + \frac{1}{4}\right)
\]
is infinite, so the corresponding series
\[
\left(1 - \frac{1}{2^2}\right) + \left( \frac{1}{3} - \frac{1}{4^2}\right) + \left( \frac{1}{5} - \frac{1}{6^2}\right) + \cdots = \sum_{n=1}^{\infty} \left( \frac{1}{2n-1} - \frac{1}{4n^2}\right)
\]
must be also.

4. Use the ratio test with \( a_n = \frac{nx^n}{n^2 - 1} = \frac{nx^n}{(n-1)(n+1)} \), so that \( a_{n+1} = \frac{(n+1)x^{n+1}}{n(n+2)} \) and we have
\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(n+1)x^{n+1}}{n(n+2)} \cdot \frac{n(n-1)}{x^n} \right|
= \lim_{n \to \infty} \left| x \frac{n^2 + n^2 - n - 1}{n^3 + 2n^2} \right| = |x|
\]
since
\[
\lim_{n \to \infty} \frac{n^3 + n^2 - n - 1}{n^3 + 2n^2} = 1
\]
by three applications of L’Hopital’s rule, or by writing
\[
\lim_{n \to \infty} \frac{n^3 + n^2 - n - 1}{n^3 + 2n^2} = \lim_{n \to \infty} \frac{1 + \frac{1}{n} - \frac{1}{n^2} - \frac{1}{n^3}}{1 + \frac{2}{n}} = \frac{1 + 0 - 0 - 0}{1 + 0} = 1.
\]
So the series converges if \(|x| < 1\), it diverges if \(|x| > 1\), and we don’t know yet what it does if \(|x| = 1\); i.e., if \(x = 1\) or \(x = -1\). At \(x = 1\) the series becomes
\[
\sum_{n=2}^{\infty} \frac{n}{n^2 - 1} > \sum_{n=2}^{\infty} \frac{n}{n^2} = \sum_{n=2}^{\infty} \frac{1}{n} = \infty \quad (p = 1),
\]
so it diverges when \(x = 1\). At \(x = -1\) it is
\[
\sum_{n=2}^{\infty} (-1)^n \frac{n}{n^2 - 1}.
\]
This alternates, and the nonalternating part is \(n/(n^2 - 1)\), which decreases to zero as \(n \to \infty\), so the series converges at \(x = -1\). Therefore
\[
\sum_{n=2}^{\infty} \frac{nx^n}{n^2 - 1} \text{ converges for } -1 < x < 1.
\]
To find the function that this series converges to, start by writing
\[ \frac{n}{n^2 - 1} = \frac{1}{2} \frac{2n}{(n + 1)(n - 1)} = \frac{1}{2} \frac{(n + 1) + (n - 1)}{(n + 1)(n - 1)} = \frac{1}{2} \left( \frac{1}{n - 1} + \frac{1}{n + 1} \right), \]
so that
\[ (A) \sum_{n=2}^{\infty} \frac{n x^n}{n^2 - 1} = \frac{1}{2} \sum_{n=2}^{\infty} \frac{x^n}{n - 1} + \frac{1}{2} \sum_{n=2}^{\infty} \frac{x^n}{n + 1}. \]

By integrating the series (G) from problem 1 we get
\[ (B) \sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln(1 - x) \text{ if } -1 \leq x < 1, \]
and the two series in (A) are slight variations on this. We can rewrite (A) as
\[ \sum_{n=2}^{\infty} \frac{n x^n}{n^2 - 1} = \frac{x}{2} \sum_{n=1}^{\infty} \frac{x^{n-1}}{n - 1} + \frac{1}{2x} \sum_{n=2}^{\infty} \frac{x^{n+1}}{n + 1} = \frac{x}{2} \sum_{n=1}^{\infty} \frac{x^n}{n} + \frac{1}{2x} \sum_{n=3}^{\infty} \frac{x^n}{n}, \]
and according to (B) this equals
\[ -\ln(1 - x) \left( \frac{x}{2} + \frac{1}{2x} \right) - \frac{1}{2x} \left( x + \frac{x^2}{2} \right) = -\left[ \frac{1}{2} + \frac{x}{4} + \frac{x^2 + 1}{2x} \ln(1 - x) \right]. \]

Therefore
\[ \sum_{n=2}^{\infty} \frac{n x^n}{n^2 - 1} = -\left[ \frac{1}{2} + \frac{x}{4} + \frac{x^2 + 1}{2x} \ln(1 - x) \right] \text{ for } -1 \leq x < 1. \]

5(a) \( \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \) and 5(b) \( \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \).

5(c) We can do this either by taking derivatives or by multiplying series together (or there is a third way, which we’ll get to in part (d)). By multiplying series:
\[ \sin x \cos x = \left( x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \right) \left( 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \right) = x - x^3 \left( \frac{1}{3!} + \frac{1}{2!} \right) + x^5 \left( \frac{1}{5!} + \frac{1}{2! 3!} + \frac{1}{4!} \right) - x^7 \left( \frac{1}{7!} + \frac{1}{5! 2!} + \frac{1}{3! 4!} + \frac{1}{6!} \right) + \ldots \]

I only asked for the first three nonzero terms, but I included the fourth one to show that there is a pattern to the coefficients. If we simplify above we get
\[ \frac{1}{3!} + \frac{1}{2!} = \frac{1}{3!} + \frac{3}{3!} = \frac{4}{3!} \quad \text{and} \quad \frac{1}{5!} + \frac{1}{2! 3!} + \frac{1}{4!} = \frac{1}{5!} + \frac{10}{5!} + \frac{5}{5!} = \frac{16}{5!}, \]
and this pattern also continues, so that
\[ \sin x \cos x = x - \frac{4x^3}{3!} + \frac{16x^5}{5!} - \frac{64x^7}{7!} + \ldots. \]
We can confirm that the coefficients really are powers of $-4$ by taking derivatives. If $f(x) = \sin x \cos x$, then

\[
    f'(x) = (\cos x) \cos x + \sin x (-\sin x) = \cos^2 x - \sin^2 x
\]

\[
    f''(x) = 2 \cos x (-\sin x) - 2 \sin x (\cos x) = -4 \sin x \cos x.
\]

Since $f''(x) = -4 f(x)$, we can easily calculate as many more derivatives as we need:

\[
    f'''(x) = -4 f'(x) = -4 (\cos^2 x - \sin^2 x), \quad f''''(x) = -4 f'''(x) = 16 \sin x \cos x,
\]

\[
    f^{(5)}(x) = -4 f''''(x) = 16 (\cos^2 x - \sin^2 x), \quad f^{(6)}(x) = -4 f^{(5)}(x) = -64 \sin x \cos x,
\]

and so on; in general

\[
    f^{(2n)}(x) = (-4)^n \sin x \cos x \quad \text{and} \quad f^{(2n+1)}(x) = (-4)^n (\cos^2 x - \sin^2 x).
\]

Since $\sin 0 = 0$ and $\cos 0 = 1$ we have

\[
    f^{(2n)}(0) = (-4)^n \sin 0 \cos 0 = 0 \quad \text{and} \quad f^{(2n+1)}(x) = (-4)^n (\cos^2 0 - \sin^2 0) = (-4)^n.
\]

Therefore

\[
    \sin x \cos x = \sum_{n=0}^{\infty} (-4)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{4x^3}{3!} + \frac{16x^5}{5!} - \frac{64x^7}{7!} + \ldots
\]

5(d) Replacing $x$ by $2x$ in part (b) we get

\[
    \sin 2x = \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1}}{(2n+1)!} x^{2n+1},
\]

\[
    = 2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \frac{128x^7}{7!} + \ldots
\]

The comment I was looking for is that this appears to be twice the series we got in part (c). In fact, if you believe (C), then it is definitely twice the series we got there, because

\[
    \frac{1}{2} \sin 2x = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1}}{(2n+1)!} x^{2n+1} = \sum_{n=0}^{\infty} (-4)^n \frac{x^{2n+1}}{(2n+1)!}.
\]

What we have here is a Taylor series proof that $\sin 2x = 2 \sin x \cos x$. If you knew this identity already, then it gives you an easy way to do part (c):

\[
    \sin x \cos x = \frac{1}{2} \sin 2x = \sum_{n=0}^{\infty} (-4)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{4x^3}{3!} + \frac{16x^5}{5!} - \ldots
\]

by (D).