Show all your work. If you use your calculator to compute an answer, you must write down enough information on what you have done that your method is understandable.

1. 

(a) (5 pts.) State the formal limit definition of \( f'(a) \).

\[
 f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}
\]

(b) (9 pts.) Briefly explain the definition in part (a). For instance, what is it intended to calculate? What is the algebraic part of the expression calculating? Why is a limit necessary? You may include a graph as part of the explanation, but must verbally explain what the graph is indicating.

(c) (8 pts.) Use the definition in part (a) to show that if \( f(x) = x^2 - x \), then \( f'(3) = 5 \).

\[
 f'(3) = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h \to 0} \frac{(3+h)^2 - (3^2) - (3^2 - 3)}{h}
\]

\[
 = \lim_{h \to 0} \frac{9 + 6h + h^2 - 3 - h - 9 + 3}{h}
\]

\[
 = \lim_{h \to 0} \frac{h(6 + h - 1)}{h}
\]

\[
 = \lim_{h \to 0} 5 + h = 5.
\]
2. (12 pts. – 6 pts. each) Give all possible correct answers to:

(a) If \( y = 2 \cos x - \frac{x^3}{3} + 3 \sqrt{x}, \) then \( \frac{dy}{dx} = -2 \sin x - x^2 + \frac{3}{2} x^{-\frac{1}{2}} = -2 \sin x - x^2 + \frac{3}{2 \sqrt{x}} \)

\( y = 2 \cos x - \frac{1}{3} x^3 + 3 \sqrt{x} \)

(b) If \( g(t) = \frac{2}{t} + 2^t - 5 \ln t, \) then \( g'(t) = -2 t^{-2} + (\ln 2) 2^t - 5 \left( \frac{1}{t} \right) = -\frac{2}{t^2} + (\ln 2) 2^t - \frac{5}{t} \)

\( g(t) = 2t^{-1} + 2^t - 5 \ln t \)

3. (12 pts. – 6 pts. each) Give all possible correct answers to:

(a) If \( \frac{dT}{dr} = r^3 + \frac{2}{r} + 3 \sin r, \) then \( T = \frac{r^4}{4} + 2 \ln r - 3 \cos r + C \)

\( T' = r^3 + 2r^2 + 3 \sin r \)

(b) If \( s''(t) = -32, \) then \( s(t) = \frac{1}{16} s^2 + C s + D \)

4. (6 pts. – 3 pts. each) Consider the differential equation \( \frac{dy}{dx} = y - x. \)

(a) Verify that \( y = Ce^x + x + 1 \) is a solution, for all values of \( C. \)

\( \frac{dy}{dx} = \frac{d}{dx} (Ce^x + x + 1) = Ce^x + 1 \)

\( y - x = (Ce^x + x + 1) - x - Ce^x + 1 \)

\( s = \frac{dy}{dx} = y - x \)

(b) Find the solution that satisfies the condition \( y(1) = 3. \)

\( 3 = y(1) = Ce^1 + 1 + 1 \)

\( 3 = Ce^1 + 2 \)

\( 1 = Ce \)

\( C = \frac{1}{e} \), so \( y = \frac{1}{e} e^x + x + 1 = e^{(x-1)} + x + 1 \)
5. (12 pts. – 4 pts. each) Suppose a bug moves on a straight line so that its position at time \( t \) seconds is given by the function
\[
s(t) = t^3 - 2t^2 + 6 \text{ cm},
\]
where positive positions are to the right. Determine:

(a) in which direction (left/right) and how quickly the bug is traveling at \( t = 1 \). (Specify units.)
\[
s'(t) = 3t^2 - 4t
\]
\[
s'(1) = 3(1)^2 - 4(1) = -1 \text{ cm/sec}
\]
so bug is moving to the left

(b) whether the bug is speeding up or slowing down at \( t = 1 \), and at what rate. (Specify units.)
\[
s''(t) = 6t - 4
\]
\[
s''(1) = 6(1) - 4 = 2 \text{ cm/sec}^2
\]
Since this is opposite in sign from \( s'(1) \), the bug is slowing down

(c) the bug’s average velocity from \( t = 0.5 \) to \( t = 1 \). (Specify units.)
\[
\frac{s(1) - s(0.5)}{1 - 0.5} = \frac{5 - 5.625}{0.5} = -1.25 \text{ cm/sec}
\]

6. (18 pts. – 2 pts. each) The graph of \( y = f(x) \) is shown. \( F(x) \) denotes an antiderivative of \( f(x) \).

In answering the following, to describe ‘where’ give either a list of \( x \)-coordinates, or intervals on the \( x \)-axis.

(a) Where does \( F \) have stationary point(s)? \(-1.5, 0.5, 1.5\)

(b) Where is \( F \) increasing? \((-\infty, -1.5) \cup (1.5, \infty)\)

(c) Where is \( F \) concave down? \((-\infty, 0) \cup (1, 1.5)\)

(d) Where does \( F \) have a local maximum(s)? \(-1.5\)

(e) Where does \( F \) have (an) inflection point(s)? \(0, 1, 1.5\)

(f) Where is \( f' \) positive? \((0, 1) \cup (1.5, \infty)\)

(g) Where is \( f \) decreasing most rapidly? \(1.25\)

(h) Which is larger, \( F(1) \) or \( F(1.5) \)? \(F(1.5)\)

(i) Which is larger, \( f'(-1) \) or \( f'(-0.5) \)? \(f'(-1)\)
7. (9 pts. – 3 pts. each) Find the exact $x$-coordinates of the following on the graph of $y = x^3 - 7x + 5$. Show enough work to justify your answers.

(a) All local maxima.

\[
y' = 3x^2 - 7 = 0
data = \frac{7}{3}
x = \pm \frac{\sqrt{21}}{3}
\]

From a graph it is clear there is a local max for a negative value of $x$, so $x = -\frac{\sqrt{21}}{3}$ must be it.

(b) All local minima.

\[
y' = 3x^2 - 7 = 0
data = \frac{7}{3}
x = \pm \frac{\sqrt{21}}{3}
\]

We could check that $y'$ changes from $+$ to $-$ at $x = -\frac{\sqrt{21}}{3}$ or that $y'' \left|_{x=-\frac{\sqrt{21}}{3}} \right. < 0$ to show this is a min on the graph.

\[
x = +\frac{\sqrt{21}}{3}\text{, by reasoning similar to (a)}
\]

(c) All inflection points.

\[
y'' = 6x, \quad y'' = 0 \text{ at } x = 0.
\]

Furthermore, $y''$ changes sign at $x = 0$ so this is an inflection point.

(We can also tell from the graph an inflection point exists, so this must be it.)

8. (9 pts. – 3 pts. each) Give short answers (a sentence or two).

(a) Can two different functions have the same rate of change? Explain.

Yes, they might just differ by a constant. E.g., $y = x^2$ and $y = x^2 - 3$.

(b) If a function $f$ has an increasing derivative, then what does that say about the graph of $f$? Explain.

The graph is concave up, since its slope gets larger as we move to the right.

(c) If $f$ has a stationary point at $x = 2$, must it have a local max or min there? Explain.

No, it might have a "saddle point."