1. Loosely speaking, this is the reciprocal of the second integral of the week, but this one is much easier: if we let \( u = x + \sin x \), then \( du = (1 + \cos x)dx \) and we have
\[
\int \frac{1 + \cos x}{x + \sin x} \, dx = \int \frac{du}{u} = \ln |u| + C = \ln |x + \sin x| + C.
\]

2(a) This is #24 in the table with \( a = 1 \):
\[
\int \frac{dx}{x^2 + 1} = \arctan x + C.
\]
(b) If we substitute \( x = \frac{1}{u} \) in the above integral then \( dx = -\frac{du}{u^2} \) and we have
\[
\int \frac{dx}{x^2 + 1} = \int \left( \arctan \left( \frac{1}{u} \right) \right) \, du = \int \frac{du}{\frac{1}{u}^2 + u^2} = -\int \frac{du}{1 + u^2}.
\]
This is the same integral as in part (a) except for the minus sign, so we conclude that
\[
\int \frac{dx}{x^2 + 1} = -\arctan u + C = -\arctan \frac{1}{x} + C.
\]
(c) Since the answers to (a) and (b) must be the same, it follows that
\[
\arctan x = -\arctan \frac{1}{x} + C
\]
for some constant \( C \). (Note: that’s what I was looking for in this part; what follows would be extra credit.) The constant isn’t exactly constant, however; there’s one constant when \( x > 0 \) and a different constant when \( x < 0 \). If we let \( x = 1 \) then we get
\[
\arctan 1 = -\arctan 1 + C \implies \frac{\pi}{4} = -\frac{\pi}{4} + C \implies C = \frac{\pi}{2},
\]
and this is also what we’d get if we let \( x \to \infty \) or \( x \to 0^+ \). But if we let \( x = -1 \) then we get
\[
\arctan(-1) = -\arctan(-1) + C \implies -\frac{\pi}{4} = \frac{\pi}{4} + C \implies C = -\frac{\pi}{2},
\]
and this is also what we’d get if we let \( x \to -\infty \) or \( x \to 0^- \). In conclusion,
\[
\arctan x + \arctan \frac{1}{x} = \begin{cases} 
\frac{\pi}{2} & \text{if } x > 0 \\
-\frac{\pi}{2} & \text{if } x < 0.
\end{cases}
\]

3(a) I hope you recognized this problem from the second lab. It becomes a lot easier if we first substitute \( x = w^2 \) to get rid of the square root. Then \( dx = 2w \, dw \) and we have
\[
(1) \quad \int e^{\sqrt{x}} \, dx = \int e^w \, 2w \, dw = 2 \int w e^w \, dw.
\]
Now this is a fairly easy integration by parts problem (alternatively, #14 in the table can be used with \( a = 1 \) and \( p(x) = x = w \)). We take \( u = 2w \) and \( dv = e^w \, dw \), so that \( du = 2 \, dw \) and \( v = e^w \) and we have
\[
(2) \quad 2 \int w e^w \, dw = 2 \left( w e^w - \int e^w \, dw \right) = 2 (w e^w - e^w) + C.
\]
Combining this with (1) we have
\[ \int e^{\sqrt{x}} \, dx = 2 \left( w \, e^w - e^w \right) + C. \]

3(b) Substitute \( x = w^3 \) to get rid of the cube root. Then \( dx = \frac{3w^2}{w} \, dw \) and we have
\[ \int e^{\sqrt[3]{x}} \, dx = \int e^{w^3} \, \frac{3w^2}{w} \, dw = 3 \int w^2 \, e^w \, dw. \]
Either use #14 in the table again, or integrate this by parts with \( u = w^2 \) and \( dv = e^w \, dw \), so that \( du = 2w \, dw \) and we have
\[ \int w^2 e^w \, dw = w^2 e^w - 2 \int w \, e^w \, dw. \]

We did the remaining integral already in part (a), so putting (2), (3) and (4) together we have
\[ \int e^{\sqrt[3]{x}} \, dx = 3 \left[ w^2 e^w - 2 \left( w \, e^w - e^w \right) \right] + C. \]

4. These integrals are all improper because the denominators become zero at the upper endpoint \( x = 1 \).
Two of them can be done by elementary methods, while the other four need the gamma function (which was mentioned in problem 38 in section 7.7). The simplest one is (vi), because that’s the only one that the natural substitution \( u = 1 - x^6 \) works nicely on. We have \( du = -6x^5 \, dx \) and
\[ \int \frac{x^5 \, dx}{\sqrt{1 - x^6}} = -\frac{1}{3} \int \frac{du}{\sqrt{u^2}} = -\frac{1}{3} \int u^{-\frac{1}{2}} \, du = -\frac{1}{3} \left( 2u^{\frac{1}{2}} \right) + C = -\frac{1}{3} \sqrt{1 - x^6} + C. \]
Therefore
\[ \int_0^1 \frac{x^5 \, dx}{\sqrt{1 - x^6}} = -\frac{1}{3} \sqrt{1 - x^6} \bigg|_0^1 = -\frac{1}{3} \left( \sqrt{1 - 1} - \sqrt{1 - 0} \right) = -\frac{1}{3} (0 - 1) = \frac{1}{3}. \]
The other nice one is (iii). If we substitute \( u = x^3 \) then \( du = 3x^2 \, dx \), so that \( \frac{du}{3} = x^2 \, dx \) and we have
\[ \int \frac{x^2 \, dx}{\sqrt{1 - x^6}} = \frac{1}{3} \int \frac{du}{\sqrt{1 - u^2}}. \]
This we can look up; it’s #28 with \( a = 1 \). Therefore
\[ \int \frac{x^2 \, dx}{\sqrt{1 - x^6}} = \frac{1}{3} \arcsin u + C = \frac{1}{3} \arcsin (x^3) + C, \]
and so
\[ \int_0^1 \frac{x^2 \, dx}{\sqrt{1 - x^6}} = \frac{1}{3} \arcsin (x^3) \bigg|_0^1 = \frac{1}{3} \left( \arcsin 1 - \arcsin 0 \right) = \frac{1}{3} (\pi/2 - 0) = \frac{\pi}{6}. \]
For the record, if we define
\[ G = \left[ \frac{\Gamma \left( \frac{1}{3} \right) \Gamma \left( \frac{2}{3} \right)}{2^{\frac{4}{3}} \pi} \right]^3 \]
with \( \Gamma(x) \) denoting the gamma function, then
and since once we know decimal places, the numerical values are 1

If we set \( A(7) = 1 \),

\[
\begin{align*}
\text{(iv) } & \frac{\pi}{6} = \int_{0}^{\infty} \frac{x^3 \, dx}{\sqrt{1-x^6}} = \int_{0}^{1} x^3 \, dx = \frac{1}{2}
\end{align*}
\]

\[
\begin{align*}
\text{(v) } & \frac{\pi}{2\sqrt{3}} = \int_{0}^{\infty} \frac{x^4 \, dx}{\sqrt{1-x^6}} = \int_{0}^{1} x^4 \, dx = \frac{1}{2\sqrt{3}}
\end{align*}
\]

If \( x \) is between 0 and 1 then 1 > \( x^2 > x^3 > x^4 > x^5 \), so (i) > (ii) > (iii) > (iv) > (v). To four decimal places, the numerical values are 1.2143, 0.7011, 0.5236, 0.4312, 0.3734, 0.3333 respectively.

5. Following the hint we look at

\[
(x+3)^2 - (x^2 + 9) = x^2 + 6x + 9 - (x^2 + 9) = 6x.
\]

Then

\[
\begin{align*}
\int_{0}^{\infty} \frac{x \, dx}{(x+3)^2(x^2+9)} &= \frac{1}{6} \int_{0}^{\infty} \frac{6x \, dx}{(x+3)^2(x^2+9)} \\
&= \frac{1}{6} \int_{0}^{\infty} \frac{(x+3)^2 - (x^2 + 9)}{(x+3)^2(x^2+9)} \, dx \\
&= \frac{1}{6} \int_{0}^{\infty} \frac{x^2 + 9}{(x+3)^2(x^2+9)} \, dx - \frac{1}{6} \int_{0}^{\infty} \frac{x^2 + 9}{(x+3)^2(x^2+9)} \, dx \\
&= \frac{1}{6} \int_{0}^{\infty} \frac{dx}{x^2 + 9} - \frac{1}{6} \int_{0}^{\infty} \frac{dx}{(x+3)^2} \\
\end{align*}
\]

Alternatively, we can try to break the integral apart using the machinery of partial fractions. We need to find constants \( A, B, C, D \) such that

\[
\frac{x}{(x+3)^2(x^2+9)} = \frac{A}{x+3} + \frac{B}{(x+3)^2} + \frac{Cx+D}{x^2+9}.
\]

Clearing fractions by multiplying through by \((x+3)^2(x^2+9)\) we get

\[
(6) \quad x = A(x+3)(x^2+9) + B(x^2+9) + (Cx+D)(x+3)^2.
\]

Unfortunately, only one of the constants is really easy to find—if we set \( x = -3 \) then we have \(-3 = B(9 + 9)\) so \( B = -\frac{1}{3} \). If we are willing to use complex numbers then we can find \( C \) and \( D \) by setting \( x = 3i \), which gives

\[
3i = (3Ci + D)(3i + 3)^2 = (3Ci + D)(9i^2 + 2i + 1) = (3Ci + D)18i, \quad \text{or} \quad \frac{1}{6} = 3Ci + D.
\]

Since \( C \) and \( D \) must be real, we must have \( D = \frac{1}{6} \) and \( C = 0 \). If we actually multiply out the right side of (6) we’ll get

\[
x = (A + C)x^3 + \text{lower degree terms in } x,
\]

so \( A + C \) must be zero (since there is no \( x^3 \) term on the left side of (6)), and if we know \( C = 0 \) then this forces \( A = 0 \).

Another sneaky way to find some of the constants is to take the derivative of (6):

\[
(7) \quad 1 = A(x+3)(2x) + A(x^2+9) + B(2x) + C(x+3)^2 + (Cx+D)(2x+3)
\]

If we set \( x = -3 \) in (7) we get \( 1 = 18A - 6B \), and since we already know \( B = -\frac{1}{3} \) this forces \( A = 0 \). As above, once we know \( A = 0 \) then we must have \( C = 0 \). We can now set \( x = 0 \) in (7), which gives \( 1 = 9A + 9C + 6D \), and since \( A \) and \( C \) are both zero this gives \( D = \frac{1}{6} \).
If we fully multiply out the right side of (6) we get
\[ x = (A + C)x^3 + (A + B + 6C + D)x^2 + (9A + 9C + 6D)x + 27A + 9B + 9D, \]
and equating coefficients of the various powers of \( x \) gives
(i) \( A + C = 0 \), 
(ii) \( A + B + 6C + D = 0 \), 
(iii) \( 9A + 9C + 6D = 1 \), 
(iv) \( 27A + 9B + 9D = 0 \).

Assuming we know already that \( B = -\frac{1}{6} \) but nothing else, we can get \( D \) by multiplying (i) by \( 9 \) and substituting in (iii), which gives \( 6D = 1 \), \( D = \frac{1}{6} \). Therefore \( B + D = 0 \), and using this in (iv) we get \( 27A = 0 \), so that \( A = 0 \). Then (i) or (ii) implies that \( C = 0 \).

Once we know \( A, B, C, D \) we have
\[
\int_{0}^{\infty} \frac{x \, dx}{(x + 3)^2(x^2 + 9)} = \int_{0}^{\infty} \left( \frac{0}{x + 3} + \frac{-\frac{1}{6}}{(x + 3)^2} + \frac{0x + \frac{1}{6}}{x^2 + 9} \right) \, dx = \frac{1}{6} \int_{0}^{\infty} \frac{dx}{x^2 + 9} - \frac{1}{6} \int_{0}^{\infty} \frac{dx}{(x + 3)^2},
\]
which was (5), so with more or less difficulty we are eventually led to (5). We can look up the first integral in (5); using #24 with \( a = 3 \) we have
\[
\frac{1}{6} \int_{0}^{\infty} \frac{dx}{x^2 + 9} = \frac{1}{6} \left( \frac{1}{3} \arctan \frac{x}{3} \right) \bigg|_{0}^{\infty} = \frac{1}{18} \left( \arctan \infty - \arctan 0 \right) = \frac{1}{18} \left( \frac{\pi}{2} - 0 \right) = \frac{\pi}{36}.
\]

The second integral in (5) we can do straightaway:
\[
\frac{1}{6} \int_{0}^{\infty} \frac{dx}{(x + 3)^2} = \frac{1}{6} \left( \frac{-1}{x + 3} \right) \bigg|_{0}^{\infty} = \frac{1}{6} \left( \frac{-1}{\infty} - \frac{-1}{3} \right) = \frac{1}{6} \left( 0 + \frac{1}{3} \right) = \frac{1}{18}.
\]

Or, if you’re not sure about this integration, you can substitute \( u = x + 3 \), so that \( du = dx \). If \( x = 0 \) then \( u = 0 + 3 = 3 \), and if \( x = \infty \) then \( u = \infty + 3 = \infty \), so this gives
\[
\frac{1}{6} \int_{0}^{\infty} \frac{dx}{(x + 3)^2} = \frac{1}{6} \int_{3}^{\infty} \frac{du}{u^2} = \frac{1}{6} \int_{3}^{\infty} u^{-2} \, du
\]
\[
= \frac{1}{6} \left( \frac{u^{-1}}{-1} \right) \bigg|_{3}^{\infty} = \frac{-1}{6u} \bigg|_{3}^{\infty}
\]
\[
= \frac{-1}{6\infty} - \frac{-1}{6 \cdot 3} = 0 + \frac{1}{18} = \frac{1}{18}.
\]

So we finally have
\[
\int_{0}^{\infty} \frac{x \, dx}{(x + 3)^2(x^2 + 9)} = \frac{1}{6} \int_{0}^{\infty} \frac{dx}{x^2 + 9} - \frac{1}{6} \int_{0}^{\infty} \frac{dx}{(x + 3)^2} = \frac{\pi}{36} - \frac{1}{18} = \frac{\pi - 2}{36}.
\]