Answer Key for Quiz 3 (section A)

1. Following the hint we look at \((x + 2)^2 - (x^2 + 4) = x^2 + 4x + 4 - x^2 - 4 = 4x\). Therefore

\[
\int \frac{x \, dx}{(x + 2)(x^2 + 4)} = \frac{1}{4} \int \frac{4x \, dx}{(x + 2)(x^2 + 4)} = \frac{1}{4} \int \frac{(x + 2)^2 - (x^2 + 4)}{(x + 2)(x^2 + 4)} \, dx = \frac{1}{4} \int \frac{(x + 2)^2 \, dx}{(x + 2)(x^2 + 4)} - \frac{1}{4} \int \frac{(x^2 + 4) \, dx}{(x + 2)(x^2 + 4)} = \frac{1}{4} \int \frac{(x + 2) \, dx}{x^2 + 4} - \frac{1}{4} \int \frac{dx}{x + 2}.
\]

The first integral can be looked up; it’s #25 in the table with \(a = 2 = c\) and \(b = 1\). Or we can break it apart:

\[
\int \frac{x \, dx}{(x + 2)(x^2 + 4)} = \frac{1}{4} \int \frac{dx}{x^2 + 4} + 1 \int \frac{x \, dx}{x^2 + 4} - \frac{1}{4} \int \frac{dx}{x + 2},
\]

and now we can look up the second integral and substitute \(u = x^2 + 4\) in the first, so that \(du = 2x \, dx\) and therefore \(\frac{du}{2} = \frac{x \, dx}{4}\). This makes the first integral into a log, and the third is also a log, so we finally have

\[
\int \frac{x \, dx}{(x + 2)(x^2 + 4)} = \frac{1}{8} \left( \ln |u| + \frac{1}{2} \left( \frac{1}{2} \arctan \frac{x}{2} \right) \right) - \frac{1}{4} \ln |x + 2| + C
= \frac{1}{8} \ln (x^2 + 4) + \frac{1}{4} \arctan \frac{x}{2} - \frac{1}{4} \ln |x + 2| + C.
\]

2. I like the second suggestion the best: if we let \(u = \sqrt{1 + t^2}\), then

\[
du = \frac{dt}{\sqrt{1 + t^2}},
\]

so

\[
\int \frac{dt}{t^2 \sqrt{1 + t^2}} = \int (-du) = -u + C = -\frac{\sqrt{1 + t^2}}{t} + C.
\]

The third suggestion is nice too, if you don’t mind a little trigonometry: if \(t = \tan \theta\) then \(dt = \sec^2 \theta \, d\theta\) and \(\sqrt{1 + t^2} = \sqrt{1 + \tan^2 \theta} = \sqrt{\sec^2 \theta} = \sec \theta\), so the integral becomes

\[
\int \frac{dt}{t^2 \sqrt{1 + t^2}} = \int \frac{\sec^2 \theta \, d\theta}{\sec^2 \theta \, \tan \theta} = \int \frac{\sec \theta \, d\theta}{\tan \theta}.
\]

This improves a lot if we multiply top and bottom by \(\cos^2 \theta\):

\[
\int \frac{\sec \theta \, d\theta}{\tan^2 \theta} = \int \frac{\sec \theta \, d\theta \cos^2 \theta}{\tan^2 \theta \cos^2 \theta} = \int \frac{\cos \theta \, d\theta}{\sin^2 \theta}.
\]

Now let \(v = \sin \theta\), so that \(dv = \cos \theta \, d\theta\) and we have

\[
\int \frac{\cos \theta \, d\theta}{\sin^2 \theta} = \int \frac{dv}{v^2} = -\frac{1}{v} + C = -\frac{1}{\sin \theta} + C.
\]
We still have to go back from $\theta$ to $t$. One way to do this is to draw the triangle implied by the substitution $t = \tan \theta$:

$$\sqrt{1 + t^2}.$$  

We see that $\sin \theta = \frac{t}{\sqrt{1 + t^2}}$, and therefore

$$\int \frac{dt}{t^2 \sqrt{1 + t^2}} = \int \frac{\cos \theta \, d\theta}{\sin^2 \theta} = -\frac{1}{\sin \theta} + C = -\frac{\sqrt{1 + t^2}}{t} + C$$

as before.

The first suggestion is less ambitious than the others, but it also works pretty well. If we let $x = \frac{1}{t} = t^{-1}$, then $dx = -\frac{1}{t^2} \, dt$, so that $-dx = \frac{dt}{t^2}$. If $x = \frac{1}{t}$ then $t = \frac{1}{x}$, so this gives

$$\int \frac{dt}{t^2 \sqrt{1 + t^2}} = \int \frac{dt}{t^2 \sqrt{1 + t^2}} = \int (-dx) \frac{1}{\sqrt{1 + \frac{1}{x^2}}} = -\int \frac{dx}{\sqrt{x^2 + 1}} = -\int \frac{x \, dx}{\sqrt{x^2 + 1}}$$

We can now either let $w = x^2 + 1$ or $y = \sqrt{x^2 + 1}$. In the first case $dw = 2x \, dx$, so $\frac{dy}{\sqrt{w}} = x \, dx$ and we have

$$\int \frac{dt}{t^2 \sqrt{1 + t^2}} = -\int \frac{x \, dx}{\sqrt{x^2 + 1}} = -\frac{1}{2} \int \frac{dw}{\sqrt{w}} = -\frac{1}{2} \left( \frac{\sqrt{w}^2}{2} \right) + C$$

This is #11 in the table with $b = 5$ and $a = 2$ (or the other way around), so

$$\int \frac{dx}{\sqrt{x^2 + 1}} = \frac{1}{2} \left[ 5 \cos 5x \cos 2x - 2 \sin 2x \cos 5x \right] + C$$

3. This is #11 in the table with $b = 5$ and $a = 2$ (or the other way around), so

$$\int \frac{dx}{\sqrt{x^2 + 1}} = \frac{1}{2} \left( 5 \cos 5x \cos 2x + 2 \sin 2x \sin 5x + C \right)$$

To do this by hand takes two integrations by parts, or some trigonometry. We omit the former to save space; the latter uses the equation we get by adding together the identities

$$\cos (5x - 2x) = \cos 5x \cos 2x + \sin 5x \sin 2x$$

$$\cos (5x + 2x) = \cos 5x \cos 2x - \sin 5x \sin 2x.$$

Thus we have

$$\int \cos 5x \cos 2x \, dx = \frac{1}{2} \int \cos 3x + \cos 7x \, dx = \frac{1}{2} \left( \frac{\sin 3x}{3} + \frac{\sin 7x}{7} \right) + C.$$