1. Let \( A = \begin{bmatrix} 7 & 12 & 2 & -17 \\ -5 & -9 & -1 & 13 \\ 2 & 1 & 3 & 0 \\ 3 & 2 & 4 & -1 \end{bmatrix} \), let \( \mathbf{c} = \begin{bmatrix} 37 \\ -26 \\ 13 \\ 19 \end{bmatrix} \) and let \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \) and \( \mathbf{v}_4 \) be the corresponding columns of \( A \).

1A: Write all the solutions of \( A\mathbf{x} = \mathbf{c} \) in the form \( \mathbf{p} + \mathbf{v}_h \) where \( \mathbf{p} \) is a particular solution of \( A\mathbf{x} = \mathbf{c} \) and \( \mathbf{v}_h \) represents all solutions of the corresponding homogeneous equation. Write down any RREF matrix you used to help solve this problem.

By calculator, the RREF of \([A|\mathbf{c}]\) is \( \begin{bmatrix} 1 & 0 & 2 & 1 & \mid & 2 \\ 0 & 1 & 1 & -1 & \mid & 2 \\ 0 & 0 & 0 & 0 & \mid & 0 \\ 0 & 0 & 0 & 0 & \mid & 0 \end{bmatrix} \) showing the solutions \( A\mathbf{x} = \mathbf{c} \) to be \( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_2 - x_4 \\ -x_3 + x_4 \\ x_3 (\text{free}) \\ x_4 (\text{free}) \end{bmatrix} \). Let \( \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u \) is another particular solution.

1B: Find another particular solution and explain how you got it. Any choices of \( x_3 \) and \( x_4 \) above give (all) solutions of \( A\mathbf{x} = \mathbf{c} \), so for example, take \( x_3 = 1 \) and \( x_4 = 1 \) to get \( \mathbf{x} = \begin{bmatrix} 7 \\ -2 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u \) is another particular solution.

1C: Let \( \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \) be any arbitrary vector in \( \mathbb{R}^4 \). What conditions (if any) must \( b_1, b_2, b_3 \) and \( b_4 \) satisfy in order for \( \mathbf{b} \) to be in the span of \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \) and \( \mathbf{v}_4 \)? Use the method discussed in class, and again, write down any RREF matrix you used to help solve this problem.

We're asking, when does \( x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 + x_3 \mathbf{v}_3 + x_4 \mathbf{v}_4 = \mathbf{b} \) have a soln? We discussed how the underlying system of eqns can be represented via a "super-augmented" matrix

\[
\begin{bmatrix}
7 & 12 & 2 & -17 & b_1 \\
-5 & -9 & -1 & 13 & b_2 \\
2 & 1 & 3 & 0 & b_3 \\
3 & 2 & 4 & 1 & b_4
\end{bmatrix}
\]

which by calculator has RREF

\[
\begin{bmatrix}
1 & 0 & 2 & 1 & 0 & 0 & 2 & -1 \\
0 & 1 & 1 & -1 & 0 & 1 & 0 & -2 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & -2
\end{bmatrix}
\]

Which tells us that the matrix \( A \) is in RREF. The column \( \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \) changes to \( \begin{bmatrix} 2b_2 - b_4 \\ -3b_2 + 2b_4 \\ b_1 + 2b_2 + b_3 + 13b_4 \\ b_1 - 13b_2 - 17b_4 \end{bmatrix} \) and thus the underlying system is consistent \( \iff \begin{cases} 0 = 37 + 22.13 - 17.19 = 37 + 286 - 323 = 0 \checkmark \\ 0 \approx -26 - 19.13 + 13.19 = -26 - 221 + 247 = 0 \checkmark \end{cases} \)

1D: Verify that the entries of \( \mathbf{c} \) do indeed satisfy these conditions; show your work.

\[
\begin{cases}
0 \approx 37 + 22.13 - 17.19 = 37 + 286 - 323 = 0 \checkmark \\
0 \approx -26 - 19.13 + 13.19 = -26 - 221 + 247 = 0 \checkmark
\end{cases}
\]

1E: Does \( \mathbb{R}^4 = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} \)? Explain your answer.

\( \text{NO, since if } \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \text{ does not satisfy both these conditions, it will not be a Linear Combination of } \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4; \text{ i.e., } \mathbf{b} \text{ will not be in Span } \{\mathbf{v}_1, \ldots, \mathbf{v}_4\} \)