1. Consider the system of equations
\[
\begin{align*}
x_1 + 2x_2 + 4x_3 &= 21 \\
4x_1 + 7x_2 + 13x_3 &= 72 \\
3x_1 + 4x_2 + 6x_3 &= 39.
\end{align*}
\]

1A. Use your calculator to find the RREF of the augmented matrix corresponding to this system and write the answer here.
\[
\begin{bmatrix}
1 & 0 & -2 & | & -3 \\
0 & 1 & 3 & | & 12 \\
0 & 0 & 0 & | & 0
\end{bmatrix}
\]

1B. What does the answer to (1A) tell us the solution to the system is? (Answer using the standard notation developed in class)
\[
\begin{align*}
x_1 &= -3 + 2x_3 \\
x_2 &= 12 - 3x_3 \\
x_3 &= \text{free}. \quad \text{(i.e. } x_3 \text{ can be chosen arbitrarily)}
\end{align*}
\]

1C. Suppose the “4” in the third equation in the above system is replaced by a “5”. What is the RREF of the augmented matrix corresponding to this new system?
\[
\begin{bmatrix}
1 & 0 & 0 & | & 5 \\
0 & 1 & 0 & | & 9 \\
0 & 0 & 1 & | & 7
\end{bmatrix}
\]

1D. What is the solution of the new system described in part (1C)? Verify that it works!
\[
\begin{align*}
x_1 &= 5 \\
x_2 &= 0 \\
x_3 &= 1
\end{align*}
\]

2. Let \( \mathbf{v}_1 = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 4 \\ 13 \\ 6 \end{bmatrix}, \text{ and } \mathbf{b} = \begin{bmatrix} 21 \\ 72 \\ 39 \end{bmatrix}.

2A. If possible, express the vector \( \mathbf{b} \) as a linear combination of the vectors \( \mathbf{v}_1, \mathbf{v}_2, \) and \( \mathbf{v}_3 \) in two different ways, or explain why this cannot be done.

The question asks us to find two different solutions \( \mathbf{x}_1 \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} + \mathbf{x}_2 \begin{bmatrix} 2 \\ 7 \\ 6 \end{bmatrix} + \mathbf{x}_3 \begin{bmatrix} 4 \\ 13 \\ 6 \end{bmatrix} = \begin{bmatrix} 21 \\ 72 \\ 39 \end{bmatrix} \)

if possible. We solved the corresponding system in (1A) above.

Letting \( x_3 = 0 \) yields \(-3\mathbf{v}_1 + 12\mathbf{v}_2 + 0\mathbf{v}_3 = \mathbf{b} \); letting \( x_3 = 1 \) gives \(-1\mathbf{v}_1 + 7\mathbf{v}_2 + 1\mathbf{v}_3 = \mathbf{b} \).

2B. Is \( \mathbf{c} = \begin{bmatrix} 22 \\ 73 \\ 40 \end{bmatrix} \) in the span of \( \mathbf{v}_1, \mathbf{v}_2, \) and \( \mathbf{v}_3 \)? Fully explain your answer. Write down any matrices and of course there are infinitely many different ways to express \( \mathbf{c} \).

The vector \( \mathbf{c} \) is in \( \text{span} \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \} \) \( \iff \) \( \mathbf{x}_1 \mathbf{v}_1 + \mathbf{x}_2 \mathbf{v}_2 + \mathbf{x}_3 \mathbf{v}_3 = \mathbf{c} \) has a solution. However, the augmented matrix corresponding to this equation has RREF \(
\begin{bmatrix}
1 & 0 & -2 & | & 0 \\
0 & 1 & 3 & | & 0 \\
0 & 0 & 0 & | & 1
\end{bmatrix}
\), showing the system corresponding to this vector equation is inconsistent, that is, it has no solution. So \( \mathbf{c} \) is not in the span \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \).