Exam #2, Math 205B (Linear Algebra)

This take-home exam is due at class time on Friday, November 16. You may consult the textbook (or any other book) and any class notes and handouts, but please do not discuss any details of this exam with anyone except me! Please sign the bottom of this sheet and turn it in with your exam. You may ask me questions about the exam, but I reserve the right to give unsatisfying answers. Please show all work.

1. (16 points) Find the complete solution of the system

\[
\begin{pmatrix}
1 & 2 & 1 & 1 & 1 \\
1 & 3 & 1 & 3 & 1 \\
2 & 2 & 1 & 2 & 5
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{pmatrix} =
\begin{pmatrix}
7 \\
8 \\
9
\end{pmatrix}
\]

2. (22 points) Find a basis for each of the four subspaces associated with the matrix

\[
A = \begin{pmatrix}
1 & 1 & 3 & 5 & 1 \\
1 & 3 & 5 & 9 & 5 \\
-1 & 5 & 3 & 7 & 11
\end{pmatrix}
\]

What is the factored form of \( A \) that displays these bases?

3. (16 points) Find an orthonormal basis for each of the four subspaces in problem 2.

4. (22 points) Let \( \vec{v}_1 = \begin{pmatrix} 3 \\ 1 \\ 1 \\ -2 \end{pmatrix} \), \( \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 3 \\ 2 \end{pmatrix} \), and \( \vec{v}_3 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 3 \end{pmatrix} \), and let \( S \) be the subspace with basis \( \{ \vec{v}_1, \vec{v}_2 \} \).

Find the matrix \( P \) that projects vectors in \( \mathbb{R}^5 \) onto \( S \), and the matrix \( R \) that reflects through \( S \). Find also the projection \( \vec{p} \) of \( \vec{v}_3 \) onto \( S \) and the reflection \( \vec{r} \) of \( \vec{v}_3 \) through \( S \). How are \( \vec{v}_3, \vec{p} \) and \( \vec{r} \) related?

5. (14 points) Explain how you can tell that \( P = \frac{1}{10} \begin{pmatrix} 7 & 4 & 1 & -2 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ -2 & 1 & 4 & 7 \end{pmatrix} \) is a projection matrix. Find a basis for the subspace \( S \) of \( \mathbb{R}^4 \) that \( P \) projects onto, and a basis for \( S^\perp \) (the orthogonal complement of \( S \)).

6. (10 points) In Math 309 one studies certain kinds of algebraic structures, of which perhaps the most important are groups. It turns out that every group can be described in terms of matrices. In this problem I want you to show that the set of all \( n \times n \) orthogonal matrices is a group. This entails 4 things: closure, associativity, identity and inverses. Since we are working with matrices the associativity is automatic, so you have to check three things:

(i) Is the \( n \times n \) identity matrix an orthogonal matrix? 
(ii) If \( Q_1 \) and \( Q_2 \) are orthogonal matrices, is \( Q_1Q_2 \) an orthogonal matrix? 
(iii) If \( Q \) is an orthogonal matrix, is \( Q^{-1} \) an orthogonal matrix?

I affirm that I did not receive help from another person in doing this exam, nor did I give help to another student in the class.

(signed) _______________________________