Exam #1, Math 205B (Linear Algebra)

This take-home exam is due at class time on Monday, October 8. (Sooner is fine.) You may consult the textbook (or any other book) and any class notes and handouts, but please do not discuss any details of this exam with anyone except me! Please sign the bottom of this sheet and turn it in with your exam. You may ask me questions about the exam, but I reserve the right to give unsatisfying answers. Please show all work (though you are encouraged to check your answers on MATLAB or a calculator).

1. (8 points) Find the angle between the vectors \( \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \) and \( \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \).

2. (18 points) Solve the system \( \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 3 \\ 1 & 9 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ 15 \end{pmatrix} \) by any method (by hand).

3. (20 points) Find \( A^{-1} \) (by hand) if \( A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix} \).

4. (30 points) (a) Find the LU factorization of \( A = \begin{pmatrix} 1 & 4 & 3 \\ 2 & 9 & 8 \\ 3 & 16 & 18 \end{pmatrix} \).

(b) Use your answer to (a) to solve \( A \vec{x} = \begin{pmatrix} 7 \\ 13 \\ 15 \end{pmatrix} \).

(c) Find \( L^{-1} \) and \( U^{-1} \). Please show work for at least one of these.

(d) Use your answers to (c) to compute \( A^{-1} \).

5. (12 points) Let \( A \) and \( B \) be square matrices of the size, say \( n \times n \) for some \( n \geq 2 \), and let \( O \) denote the \( n \times n \) zero matrix (all of the entries of \( O \) are zero). Assume throughout this problem that \( AB = O \).

(i) Show by an example that \( BA \) need not equal \( O \). (You should be able to find a \( 2 \times 2 \) example, though you are welcome to give a larger example instead.)

(ii) If \( A \) is invertible, must \( BA = O \)? Explain. What if \( B \) is invertible?

(iii) If \( A \) and \( B \) are symmetric (i.e., \( A = A^T \) and \( B = B^T \)), prove that \( BA = O \).

6. (12 points) Suppose that \( A \) is a \( 2 \times 2 \) matrix such that \( A^{-1} = A^T \). Prove that \( A \) must be either a rotation matrix or a reflection matrix.

I affirm that I did not receive help from another person in doing this exam, nor did I give help to another student in the class.

(signed) ________________________________