Answer Key for Quiz 2 (section A)

1. We can let \( u = \ln(\ln x) \) and \( dv = \frac{dx}{x} \), in which case \( v = \ln x \) (since \( x \) is positive) and \( du = \frac{1}{\ln x} \frac{1}{x} \, dx \).

Then

\[
\int \ln (\ln x) \frac{1}{x} \, dx = \ln(\ln x) \ln x - \int \frac{dx}{x \ln x}
\]

\[
= \ln(\ln x) \ln x - \int \frac{dx}{x}
\]

\[
= \ln(\ln x) \ln x - \ln x + C = (\ln x) [\ln(ln x) - 1] + C.
\]

Or we can start by substituting \( w = \ln x \), in which case \( dw = \frac{1}{x} \, dx \) and we have

\[
\int \ln (\ln x) \frac{1}{x} \, dx = \int \ln w \, dw.
\]

This one we’ve done before: integrate by parts with \( u = \ln w \), so that \( du = \frac{1}{w} \, dw \), and we have

\[
\int \ln (\ln x) \frac{1}{x} \, dx = \int \ln w \, dw = w \ln w - \int \frac{dw}{w} = w \ln w - \ln w + C = \ln x \ln(\ln x) - \ln x + C.
\]

2. In both of 
\[
\int \text{arcsec} \, x \, dx \quad \text{and} \quad \int x \, \text{arcsec} \, x \, dx
\]

we want to integrate by parts with \( u = \text{arcsec} x \), so that \( du = \frac{1}{x \sqrt{x^2 - 1}} \, dx \). In the first case \( dv = dx \), so \( v = x \) and we have

\[
\int \text{arcsec} \, x \, dx = x \, \text{arcsec} x - \int x \frac{dx}{x \sqrt{x^2 - 1}} = x \, \text{arcsec} x - \int \frac{dx}{\sqrt{x^2 - 1}}.
\]

In the second case \( dv = x \, dx \), so that \( v = \frac{x^2}{2} \) and we have

\[
\int x \, \text{arcsec} \, x \, dx = \frac{x^2}{2} \, \text{arcsec} x - \int \frac{x^2}{2} \frac{dx}{x \sqrt{x^2 - 1}} = \frac{x^2}{2} \, \text{arcsec} x - \frac{1}{2} \int \frac{x \, dx}{\sqrt{x^2 - 1}}.
\]

This is easier than the first case, because the substitution \( w = x^2 - 1 \) will work here. We have \( dw = 2x \, dx \), so \( \frac{dw}{2} = x \, dx \) and we get

\[
\int x \, \text{arcsec} \, x \, dx = \frac{x^2}{2} \, \text{arcsec} x - \frac{1}{2} \int \frac{x \, dx}{\sqrt{x^2 - 1}}
\]

\[
= \frac{x^2}{2} \, \text{arcsec} x - \frac{1}{2} \int \frac{dw}{\sqrt{w}}
\]

\[
= \frac{x^2}{2} \, \text{arcsec} x - \frac{1}{4} \int w^{-\frac{1}{2}} \, dw
\]

\[
= \frac{x^2}{2} \, \text{arcsec} x - \frac{1}{4} 2w^{\frac{1}{2}} + C
\]

\[
= \frac{x^2}{2} \, \text{arcsec} x - \frac{1}{2} \sqrt{x^2 - 1} + C.
\]
The substitution $y = \sqrt{x^2 - 1}$ is even a little better. Then

$$dy = \frac{1}{2}(x^2 - 1)^{-\frac{1}{2}} \frac{2x}{\sqrt{x^2 - 1}} \, dx = \frac{x \, dx}{\sqrt{x^2 - 1}},$$

so

$$\int x \text{arcsec} \, x \, dx = \frac{x^2}{2} \text{arcsec} \, x - \frac{1}{2} \int x \, dx = \frac{x^2}{2} \text{arcsec} \, x - \frac{y}{2} + C$$

$$= \frac{x^2}{2} \text{arcsec} \, x - \frac{1}{2} \sqrt{x^2 - 1} + C.$$

The same substitution works in the first case, but in a very tricky way. We saw above that

$$\text{if } y = \sqrt{x^2 - 1}, \text{ then } dy = \frac{x \, dx}{\sqrt{x^2 - 1}} = \frac{x \, dx}{y}, \text{ or } \frac{dy}{x} = \frac{dx}{y}.$$  

A property of fractions which is not so well known, although easy to prove, is that

$$\text{if } \frac{a}{b} = \frac{c}{d}, \text{ then } \frac{a}{b} = \frac{c}{d} = \frac{a + c}{b + d}.$$  

Therefore

$$\int \frac{dx}{\sqrt{x^2 - 1}} = \int \frac{dx}{y} = \int \frac{dx + dy}{x + y}$$

with $y = \sqrt{x^2 - 1}$ as above. If we now let $z = x + y$, then $dz = dx + dy$ and we get

$$\int \frac{dx}{\sqrt{x^2 - 1}} = \int \frac{dx + dy}{x + y} = \int \frac{dz}{z} = \ln|z| + C = \ln|x + y| + C = \ln|x + \sqrt{x^2 - 1}| + C.$$  

This is a difficult integral, unless you happen to know that it equals a function we haven’t discussed, the inverse hyperbolic cosine of $x$. Another way of doing it is to let $x = \sec \theta$, in which case $dx = \sec \theta \tan \theta \, d\theta$ and we have

$$\int \sec \theta \tan \theta \, d\theta = \int \sec \theta \, d\theta = \ln|\sec \theta + \tan \theta| + C$$

$$= \ln|x + \sqrt{x^2 - 1}| + C.$$  

If we can do this integral, then we can say

$$\int \text{arcsec} \, x \, dx = x \text{arcsec} \, x - \int \frac{dx}{\sqrt{x^2 - 1}}$$

$$= x \text{arcsec} \, x - \ln\left(x + \sqrt{x^2 - 1}\right) + C,$$  

where we can drop the absolute values since we assumed at the start that $x$ is positive.