Answer Key for Quiz 1 (section A)

1. If we substitute $u = x^4 + x^2 + 1$ then $du = (4x^3 + 2x)dx = 2x(2x^2 + 1)dx$, so $\frac{du}{2} = (2x^2 + 1)x\,dx$ and we have

$$
\int (x^4 + x^2 + 1)^7 (2x^2 + 1)\,x\,dx = \int u^7 \frac{du}{2} = \frac{1}{2} \int u^7 \,du = \frac{1}{8} u^8 + C = \frac{1}{16} (x^4 + x^2 + 1)^8 + C.
$$

The substitution $v = x^2$ isn’t powerful enough to knock off the original integral, but it does simplify it, which is always worthwhile: $dv = 2x\,dx$, so $\frac{dv}{2} = x\,dx$ and we have

$$
\int (x^4 + x^2 + 1)^7 (2x^2 + 1)\,x\,dx = \frac{1}{2} \int (v^2 + v + 1)^7 (2v + 1)dv.
$$

Now letting $w = v^2 + v + 1$, in which case $dw = (2v + 1)dv$, gives us

$$
\int (x^4 + x^2 + 1)^7 (2x^2 + 1)\,x\,dx = \frac{1}{2} \int w^7 \,dw = \frac{w^8}{16} + C = \frac{1}{16} (x^4 + x^2 + 1)^8 + C
$$
as before.

2. (i) Because $\sin^2 \theta + \cos^2 \theta = 1$, we can write

$$
\int \frac{d\theta}{\sin \theta \cos \theta} = \int \frac{1 \cdot d\theta}{\sin \theta \cos \theta} = \int \frac{(\sin^2 \theta + \cos^2 \theta) d\theta}{\sin \theta \cos \theta}.
$$

Then we can split this up and simplify:

$$
\int \frac{(\sin^2 \theta + \cos^2 \theta) d\theta}{\sin \theta \cos \theta} = \int \frac{\sin^2 \theta \,d\theta}{\sin \theta \cos \theta} + \int \frac{\cos^2 \theta \,d\theta}{\sin \theta \cos \theta} = \int \frac{\sin \theta \,d\theta}{\cos \theta} + \int \frac{\cos \theta \,d\theta}{\sin \theta}.
$$

(ii) We can do these last two integrals by substituting $u = \cos \theta$ in the first one, and $v = \sin \theta$ in the second one. We have $du = -\sin \theta \,d\theta$ and $dv = \cos \theta \,d\theta$ respectively, so

$$
\int \frac{\sin \theta \,d\theta}{\cos \theta} = \int \frac{-du}{u} = -\ln |u| + C = -\ln |\cos \theta| + C
$$
and

$$
\int \frac{\cos \theta \,d\theta}{\sin \theta} = \int \frac{dv}{v} = \ln |v| + C = \ln |\sin \theta| + C.
$$

Therefore

$$
\int \frac{d\theta}{\sin \theta \cos \theta} = \ln |\sin \theta| + C - \ln |\cos \theta| + C,
$$
and if we want to we can also write this as

$$
\int \frac{d\theta}{\sin \theta \cos \theta} = \ln |\tan \theta| + C.
$$

There are several other ways to do the integral, of which I’ll show you only one:

$$
\int \frac{d\theta}{\sin \theta \cos \theta} = \int \frac{d\theta}{\sin \theta \cos^2 \theta} = \int \frac{\sec^2 \theta \,d\theta}{\tan \theta},
$$

where $\sec \theta$ is the reciprocal of $\cos \theta$. (One would not learn from our textbook that there is such a function as $\sec \theta$, but some of you may have picked it up on the street somewhere.) If we now let $w = \tan \theta$, then $dw = \sec^2 \theta \,d\theta$ and we have

$$
\int \frac{d\theta}{\sin \theta \cos \theta} = \int \sec^2 \theta \,d\theta = \int \frac{dw}{w} = \ln |w| + C = \ln |\tan \theta| + C
$$
as before.