1. If \( y = ax^8 + bx^4 + cx^2 \), then
\[
\frac{dy}{dx} = 8ax^7 + 4bx^3 + 2cx
\] and
\[
\frac{d^2y}{dx^2} = 56ax^6 + 12bx^2 + 2c.
\]
Plugging all this into the differential equation \( x^2 \frac{d^2y}{dx^2} - 5x \frac{dy}{dx} + 8y = x^8 \) we get
\[
x^8 = x^2 (56ax^6 + 12bx^2 + 2c) - 5x (8ax^7 + 4bx^3 + 2cx) + 8 (ax^8 + bx^4 + cx^2)
= 56ax^8 + 12bx^4 + 2cx^2 - 40ax^8 - 20bx^4 - 10cx^2 + 8ax^8 + 8bx^4 + 8cx^2
= 24ax^8.
\]
Since this is supposed to hold for all \( x \), we must have \( a = 1/24 \). There are no restrictions on \( b \) and \( c \) since all those terms cancelled. In fact, every solution of this differential equation has the form
\[
y = \frac{1}{24} x^8 + bx^4 + cx^2
\]
for some constants \( b \) and \( c \).

2. Since the density depends on the distance from Main Street, we have to slice the city along “streets” parallel to Main Street. A street \( x \) miles from Main Street has length \( 4(1 - x) \) by similar triangles.

If the street has thickness \( dx \), then the number of people living on it is
\[
density \times \text{area} = 5000e^{1-x} 4(1-x) dx = 20000(1-x)e^{1-x} dx.
\]
The distance from Main Street varies between 0 and 1, but in two different directions. The population of the city north of Main Street is
\[
P_{\text{north}} = 20000 \int_0^1 (1-x)e^{1-x} dx,
\]
and the population south of main street is exactly the same, so the total population of the city is
\[
P = 40000 \int_0^1 (1-x)e^{1-x} dx.
\]
To evaluate this we can first substitute \( w = 1 - x \), so that \( dw = -dx \) and hence \( -dw = dx \). If \( x = 0 \) then \( w = 1 - 0 = 1 \), and if \( x = 1 \) then \( w = 1 - 1 = 0 \), so we get
\[
P = 40000 \int_0^1 (1-x)e^{1-x} dx = -40000 \int_1^0 we^w dw = 40000 \int_0^1 we^w dw.
\]
Now integrate by parts with \( dv = e^w dw \) and \( u = w \), so that \( du = dw \) and \( v = e^w \) and we have
\[
P = 40000 \int_0^1 we^w dw = 40000 \left( we^w \big|_0^1 - \int_0^1 e^w dw \right)
= 40000 \left( w - 1 \right) e^w \big|_0^1
= 40000 \left[ (1-1)e^1 - (0-1)e^0 \right] = 40000.