Answer Key for Quiz 7 (section A)

1. Since $0 = x(1 - x)$ implies that either $x = 0$ or $x = 1$, the curve $y = x(1 - x)$ intersects the $x$-axis at $(0,0)$ and $(1,0)$, and is above the $x$-axis between these points:

A vertical strip of this region at a point $x$ generates a circular disk of radius $x(1 - x)$ when revolved around the $x$-axis. Since the area of such a disk is $\pi x^2(1 - x)^2$, the volume we get is obtained by multiplying by $dx$ and integrating over the possible values of $x$, namely $0 \leq x \leq 1$. This gives

$$V = \int_0^1 \pi x^2(1 - x)^2 \, dx.$$ 

No substitution helps this much, so we just multiply it out:

$$V = \pi \int_0^1 x^2 (1 - 2x + x^2) \, dx$$

$$= \pi \int_0^1 (x^2 - 2x^3 + x^4) \, dx$$

$$= \pi \left[ \frac{x^3}{3} - 2 \frac{x^4}{4} + \frac{x^5}{5} \right]_0^1$$

$$= \pi \left( \frac{1}{3} - \frac{1}{2} + \frac{1}{5} \right) = \frac{\pi}{30} (10 - 15 + 6) = \frac{\pi}{30}.$$

To do this with cylindrical shells is possible, but more difficult. A horizontal strip of the region generates a hollow cylinder of radius $y$ and some height when revolved around the $x$-axis, and to get the height takes some work. We have

$$y = x - x^2 = \frac{1}{4} = \left( x - \frac{1}{2} \right)^2 \quad \text{so} \quad \left( x - \frac{1}{2} \right)^2 = \frac{1}{4} - y \quad \text{and therefore} \quad x = \frac{1}{2} \pm \sqrt{\frac{1}{4} - y}.$$

The horizontal strips go from the left half of the parabola to the right half, so the one at a particular $y$ has height

$$h = \frac{1}{2} + \sqrt{\frac{1}{4} - y} - \left( \frac{1}{2} - \sqrt{\frac{1}{4} - y} \right) = 2 \sqrt{\frac{1}{4} - y}.$$

Then the cylindrical shells have area

$$2\pi rh = 2\pi y \left( 2 \sqrt{\frac{1}{4} - y} \right).$$

Multiplying by $dy$ and integrating over the possible $y$'s we get

$$V = 4\pi \int_0^1 y \sqrt{\frac{1}{4} - y} \, dy.$$
To do this it helps to substitute \( u = \frac{1}{4} - y \), so that \( du = -dy \) and therefore \( dy = -du \). If \( y = 0 \) then \( u = \frac{1}{4} = 0 = \frac{1}{4} \), and if \( y = \frac{1}{4} \) then \( u = \frac{1}{4} - \frac{1}{4} = 0 \). Also, if \( u = \frac{1}{4} - y \) then \( y = \frac{1}{4} - u \), so the integral becomes

\[
V = 4\pi \int_{\frac{1}{4}}^{0} \left( \frac{1}{4} - u \right) \sqrt{u} (-du) = 4\pi \int_{0}^{\frac{1}{4}} \left( \frac{1}{4} - u \right) u^{\frac{1}{2}} du \\
= \pi \int_{0}^{\frac{1}{4}} \left( \frac{1}{4} - 4u^{3} \right) du = \pi \left( \frac{2}{3} u^{\frac{3}{2}} - \frac{8}{5} u^{\frac{5}{2}} \right) \bigg|_{0}^{\frac{1}{4}} \\
= \pi \left( \frac{2}{3} \frac{1}{8} - \frac{8}{5} \frac{1}{32} \right) = \pi \left( \frac{1}{12} - \frac{1}{20} \right) = \frac{\pi}{60} (5 - 3) = \frac{\pi}{30}.
\]

2. The curves \( y = x^{1/4} \) and \( y = x^{4} \) intersect at \((0,0)\) and \((1,1)\), so the region of interest looks like

A vertical strip at a given \( x \) generates a washer with outer radius \( x^{1/4} \) and inner radius \( x^{4} \) when revolved around the \( x \)-axis. The area of such a washer is

\[
\pi \left( x^{\frac{1}{4}} \right)^{2} - \pi \left( x^{4} \right)^{2} = \pi \left( x^{\frac{1}{4}} - x^{4} \right).
\]

Multiplying by \( dx \) and integrating over all the possible values of \( x \) we get

\[
V = \pi \int_{0}^{1} \left( x^{\frac{1}{4}} - x^{4} \right) dx \\
= \pi \left( \frac{2}{3} x^{\frac{5}{4}} - \frac{1}{9} x^{5} \right) \bigg|_{0}^{1} \\
= \pi \left( \frac{2}{3} - \frac{1}{9} \right) = \frac{5\pi}{9}.
\]

This also works about as well with cylindrical shells. For a given \( y \), a horizontal strip generates a hollow cylinder with radius \( y \) and height \( y^{1/4} - y^{4} \), since if \( y = x^{1/4} \) then \( x = y^{4} \) and vice versa. The surface area of such a shell is

\[
2\pi rh = 2\pi y \left( y^{\frac{3}{4}} - y^{4} \right).
\]

Multiplying by \( dy \) and integrating over all the possible values of \( y \) we get

\[
V = 2\pi \int_{0}^{1} y \left( y^{\frac{3}{4}} - y^{4} \right) dy \\
= 2\pi \int_{0}^{1} \left( y^{\frac{7}{4}} - y^{5} \right) dy \\
= 2\pi \left( \frac{4}{9} y^{\frac{11}{4}} - \frac{1}{6} y^{6} \right) \bigg|_{0}^{1} \\
= 2\pi \left( \frac{4}{9} - \frac{1}{6} \right) = 2\pi \left( \frac{8}{18} - \frac{3}{18} \right) = \frac{5\pi}{9}.
\]