1.

If \( a_n = (-1)^n \frac{x^{2n+1}}{2n+1} \), then \( a_{n+1} = (-1)^{n+1} \frac{x^{2(n+1)+1}}{2(n+1)+1} = (-1)^{n+1} \frac{x^{2n+3}}{2n+3} \).

We could actually ignore the minus signs at this point, because the absolute values in the ratio test will get rid of them. We have to calculate

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{(-1)^{n+1} \frac{x^{2n+3}}{2n+1}}{(-1)^n \frac{x^{2n+1}}{2n+1}} = \lim_{n \to \infty} \frac{(-1)^{n+1} x^{2n+3}}{-x^{2n+1}} 2n + 1 = \frac{-x^2}{2n+3}.
\]

where to do the remaining limit we can either write

\[
\lim_{n \to \infty} \frac{2n + 1}{2n + 3} = \lim_{n \to \infty} \frac{2 + \frac{1}{n}}{2 + \frac{3}{n}} = \frac{2 + 0}{2 + 0} = 1
\]

or

\[
\lim_{n \to \infty} \frac{2n + 1}{2n + 3} = \lim_{n \to \infty} \frac{2n + 3 - 2}{2n + 1} = \lim_{n \to \infty} 1 - \frac{2}{2n + 1} = 1 - 0 = 1
\]

or use L’Hôpital’s rule, since it has the form \( \frac{\infty}{\infty} \):

\[
\lim_{n \to \infty} \frac{2n + 1}{2n + 3} \LH \lim_{n \to \infty} \frac{2}{2} = 1.
\]

So the series converges if \( x^2 < 1 \), it diverges if \( x^2 > 1 \), and we don’t know yet what it does if \( x^2 = 1 \). If \( x^2 = 1 \) then \( x = 1 \) or \( x = -1 \), so these are the endpoints that it converges in between. If we plug \( x = 1 \) into the series we get

\[
\sum_{n=0}^{\infty} (-1)^n \frac{1^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1},
\]

which converges by the alternating series test since the non-alternating part, \( \frac{1}{2n+1} \), decreases to zero as \( n \to \infty \). If we plug \( x = -1 \) into the series we get the negative of the series we get at \( x = 1 \), because \((-1)^{2n+1} = -1 \) since the power of \(-1\) is always odd. In other words, we get

\[
\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1},
\]

which again converges by the alternating series test. So the interval of convergence is \(-1 \leq x \leq 1 \). To get the function that the series converges to, calculate the derivative:

\[
\sum_{n=0}^{\infty} (-1)^n (2n + 1) \frac{x^{2n}}{2n + 1} = \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + \ldots
\]

This is geometric with ratio \(-x^2\) since each term is \(-x^2\) times the preceding term. Since the ratio test gave us \( x^2 < 1 \), we know that this series converges, and its sum is \( 1/(1 + x^2) \). Therefore the original series must be the integral of this; that is,

\[
\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n + 1} = \int \frac{dx}{1 + x^2} = \arctan x + C.
\]
If we set \( x = 0 \) above we get 0 = arctan 0 + \( C \) = 0 + \( C \) and therefore \( C = 0 \). So we finally have

\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = \arctan x \quad \text{if} \quad -1 \leq x \leq 1.
\]

This says something very interesting when \( x = 1 \):

\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = \arctan 1 = \frac{\pi}{4};
\]

that is,

\[
\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \ldots
\]

This series was known to Leibniz in the 1600s and to Gregory slightly before him. Ranjan Roy of Beloit College has pointed out that it was also found by an Indian mathematician named Madhava in the 1300s.

2. The function \( \frac{e^x - e^{-x}}{2} \) has a name; it is called the hyperbolic sine of \( x \) and denoted by \( \sinh x \). Its derivative is

\[
\frac{1}{2} \left\{ e^x - e^{-x} \right\} = \frac{e^x + e^{-x}}{2}
\]

and this function also has a name; it is called the hyperbolic cosine of \( x \) and denoted by \( \cosh x \). Its derivative is

\[
\frac{1}{2} \left\{ e^x + e^{-x} \right\} = \frac{e^x - e^{-x}}{2} = \sinh x.
\]

In other words, the derivatives of these functions repeat in a cycle of 2; so they are better than \( \sin x \) and \( \cos x \) in this sense in that the trig functions repeat in a cycle of 4. If \( f(x) = \sinh x \), then

\[
f(x) = \sinh x = f''(x) = f^{(4)}(x) = f^{(8)}(x) = f^{(12)}(x) = \ldots
\]

and

\[
f'(x) = \cosh x = f^{(5)}(x) = f^{(7)}(x) = f^{(9)}(x) = f^{(11)}(x) = f^{(13)}(x) = \ldots
\]

Since

\[
\cosh 0 = \frac{e^0 + e^0}{2} = \frac{1 + 1}{2} = 1 \quad \text{and} \quad \sinh 0 = \frac{e^0 - e^0}{2} = \frac{1 - 1}{2} = 0,
\]

we have

\[
\frac{e^x - e^{-x}}{2} = 0 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + 0 + \frac{x^7}{7!} + 0 + \frac{x^9}{9!} + 0 + \ldots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \sinh x.
\]

We also have

\[
\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = \cosh x.
\]