Answer Key for Quiz 5 (section B)

1. This is an alternating series, because of the \((-1)^{n-1}\). The non-alternating part is

\[
\frac{1}{4n^3 - n} = \frac{1}{n(2n - 1)(2n + 1)},
\]

which decreases to zero as \(n \to \infty\). Therefore the series converges by the alternating series test.

The ratio test does not work on this series because the limit is 1, which is the inconclusive case, but we could use an absolute convergence argument:

\[
\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{4n^3 - n} \right| = \sum_{n=1}^{\infty} \frac{1}{4n^3 - n} < \sum_{n=1}^{\infty} \frac{1}{4n^3} = \frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{n^3} < \infty
\]

since it is \(\frac{1}{3}\) of a \(p\)-series with \(p = 3\). So the given series converges, since it converges absolutely. It is also possible to find the number that it converges to. I’ll put this on at the end.

2. Just looking at the dominant terms, this series resembles

\[
\sum_{n=1}^{\infty} \frac{n^2}{4n^3} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n},
\]

which diverges since it is \(\frac{1}{4}\) of the \(p\)-series with \(p = 1\), so we expect that the given series probably diverges.

We actually have

\[
\sum_{n=1}^{\infty} \frac{n^2}{4n^3} > \sum_{n=1}^{\infty} \frac{n^2}{4n^3} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n} = \infty,
\]

since throwing away the 4 in the numerator and the \(n\) in the denominator makes the numerator smaller and the denominator bigger, so the given series certainly diverges.

The ratio test does not work on this series because the limit is 1, which is the inconclusive case, but we could use the integral test. Break the integral apart (we omit the details) to get

\[
\int_{1}^{\infty} \frac{x^2 + 4}{x(2x-1)(2x+1)} \, dx = \int_{1}^{\infty} \frac{x \, dx}{2x^2 - 1} + 4 \int_{1}^{\infty} \left( \frac{1}{2x} - \frac{1}{2x-1} \right) \, dx
\]

\[
= \frac{1}{8} \ln \left( 4x^2 - 1 \right) \bigg|_{1}^{\infty} + 2 \ln \left( \frac{2x - 1}{2x} \right) \bigg|_{1}^{\infty} - 2 \ln \left( \frac{2x}{2x + 1} \right) \bigg|_{1}^{\infty}
\]

\[
= \frac{1}{8} \ln \left( 4x^2 - 1 \right) \bigg|_{1}^{\infty} + 2 \ln \left( 1 - \frac{1}{4x^2} \right) \bigg|_{1}^{\infty}
\]

and this is infinite, since the first term is infinite and the second is finite. Therefore the corresponding sum must be infinite since the function is positive and (at least eventually) decreasing.

3. The presence of factorials strongly suggests that we should use the ratio test.

If \(a_n = \frac{(n!)^3}{(3n)!}\) then \(a_{n+1} = \frac{[(n+1)!]^3}{(3(n+1))!} = \frac{[(n+1)!]^3}{(3n+3)!}\).

Since \(a_n\) is positive, we don’t need the absolute values in the statement of the ratio test, so we need to calculate

\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{[(n+1)!]^3 (3n)!}{(3n+3)! (n!)^3}
\]

\[
= \lim_{n \to \infty} \frac{[(n+1)]^3 (3n)!}{(n!)^3 (3n+3)!}
\]

\[
= \lim_{n \to \infty} \frac{(n+1)! (n+1)! (n+1)!}{n! n! n!} \frac{(3n)!}{(3n+3)!}.
\]
We know that
\[
\frac{(n+1)!}{n!} = \frac{1 \cdot 2 \cdot 3 \cdots (n)(n+1)}{1 \cdot 2 \cdot 3 \cdots (n)} = n+1,
\]
and similarly
\[
\frac{(3n)!}{(3n+3)!} = \frac{1 \cdot 2 \cdot 3 \cdots (3n)}{1 \cdot 2 \cdot 3 \cdots (3n+1)(3n+2)(3n+3)} = \frac{1}{(3n+1)(3n+2)(3n+3)}.
\]
Therefore
\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{(n+1)!}{n!} \cdot \frac{(n+1)!}{n!} \cdot \frac{(3n)!}{(3n+3)!} = \lim_{n \to \infty} \frac{(n+1)(n+1)}{(3n+1)(3n+2)(3n+3)}
\]
By L’Hopital’s rule
\[
\lim_{n \to \infty} \frac{n+1}{3n+1} \Rightarrow \lim_{n \to \infty} \frac{1}{3} = \frac{1}{3} = \lim_{n \to \infty} \frac{n+1}{3n+2}
\]
so
\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{(n+1)(n+1)}{3(3n+1)(3n+2)} = \frac{1 \cdot 1}{3 \cdot 3} = \frac{1}{27}
\]
Since this is less than 1, the series converges.

To find the exact value of the series in problem 1 we can break the general term into partial fractions:
\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{4n^2 - n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(2n-1)(2n+1)}
\]
\[
= \sum_{n=1}^{\infty} (-1)^{n-1} \left( \frac{1}{2n-1} - \frac{1}{n} + \frac{1}{2n+1} \right)
\]
At this point it is probably simplest to start writing out the terms:
\[
\sum_{n=1}^{\infty} (-1)^{n-1} \left( \frac{1}{2n-1} - \frac{1}{n} + \frac{1}{2n+1} \right)
\]
\[
= 1 - 1 + \frac{1}{3} - \left( \frac{1}{3} - \frac{1}{2} + \frac{1}{5} \right) + \frac{1}{5} - \frac{1}{3} + \frac{1}{7} - \left( \frac{1}{7} - \frac{1}{4} + \frac{1}{9} \right) + \frac{1}{9} - \frac{1}{5} + \frac{1}{11} - \left( \frac{1}{11} - \frac{1}{6} + \frac{1}{13} \right) \ldots
\]
Note that a lot of terms cancel—the first two, the next two, the sixth and seventh, the ninth and tenth, and so on. The terms that don’t cancel are
\[
\sum_{n=1}^{\infty} (-1)^{n-1} \left( \frac{1}{2n-1} - \frac{1}{n} + \frac{1}{2n+1} \right) = \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \ldots
\]
\[
= 1 - \left( \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots \right)
\]
\[
= 1 - \ln 2.
\]