Answer Key for Quiz 3 (section B)

1. Following the hint we look at

\[(x + 2)^2 - (x + 1)(x + 3) = x^2 + 4x + 4 - (x^2 + 4x + 3) = 1.\]

Therefore

\[
\int \frac{dx}{(x + 1)(x + 2)^2(x + 3)} = \int \frac{(x + 2)^2 - (x + 1)(x + 3)}{(x + 1)(x + 2)^2(x + 3)} \, dx \\
= \int \frac{(x + 2)^2 \, dx}{(x + 1)(x + 2)^2(x + 3)} - \int \frac{(x + 1)(x + 3) \, dx}{(x + 1)(x + 2)^2(x + 3)} \\
= \int \frac{dx}{(x + 1)(x + 3)} - \int \frac{dx}{(x + 2)^2}.
\]

In the second integral we can let \(u = x + 2\), so that \(du = dx\) and we just have to integrate a power function. The first integral can be looked up in the table (#26 with \(a = -1\) and \(b = -3\)), or we can substitute \(v = \frac{x + 1}{x + 3}\), or we can use the same sort of trick as above on it:

\[
\int \frac{dx}{(x + 1)(x + 2)^2(x + 3)} = \frac{1}{2} \int \frac{2 \, dx}{(x + 1)(x + 3)} = \frac{1}{2} \int \frac{(x + 3) - (x + 1)}{(x + 1)(x + 3)} \, dx \\
= \frac{1}{2} \int (x + 3) \, dx - \frac{1}{2} \int (x + 1) \, dx = \frac{1}{2} \int \frac{dx}{x + 1} - \frac{1}{2} \int \frac{dx}{x + 3}.
\]

So we finally have

\[
\int \frac{dx}{(x + 1)(x + 2)^2(x + 3)} = \frac{1}{2} \int \frac{dx}{x + 1} - \frac{1}{2} \int \frac{dx}{x + 3} - \int u^{-2} \, du \quad \text{where} \quad u = x + 2 \\\n= \frac{1}{2} \ln |x + 1| - \frac{1}{2} \ln |x + 3| + \frac{1}{u} + C \\
= \frac{1}{2} \ln \left| \frac{x + 1}{x + 3} \right| + \frac{1}{x + 2} + C.
\]

2. Following the first hint, let \(x = \sin^2 \theta\), so that \(dx = 2 \sin \theta \cos \theta \, d\theta\). The motivation for this substitution is that it ought to simplify the radical:

\[
\sqrt{x(1-x)} = \sqrt{\sin^2 \theta \left(1 - \sin^2 \theta\right)} = \sqrt{\sin^2 \theta \cos^2 \theta} = \sin \theta \cos \theta.
\]

Therefore

\[
\int \frac{dx}{\sqrt{x(1-x)}} = \int \frac{2 \sin \theta \cos \theta \, d\theta}{\sin \theta \cos \theta} = \int 2 \, d\theta = 2\theta + C.
\]

Now we just have to reverse the substitution: if \(x = \sin^2 \theta\) then \(\sqrt{x} = \sin \theta\), so \(\theta = \arcsin \sqrt{x}\) and we finally have

\[
\int \frac{dx}{\sqrt{x(1-x)}} = 2\theta + C = 2 \arcsin \sqrt{x} + C.
\]

Following the second hint instead we try \(u = \sqrt{x} = x^{\frac{1}{2}}\). Then \(du = \frac{1}{2} x^{-\frac{1}{2}} \, dx = \frac{dx}{2\sqrt{x}}\), or \(2 \, du = \frac{dx}{\sqrt{x}}\). If \(u = \sqrt{x}\) then \(x = u^2\), so the integral becomes

\[
\int \frac{dx}{\sqrt{x(1-x)}} = \int \frac{dx}{\sqrt{x} \sqrt{1-x}} = \int \frac{2 \, du}{\sqrt{1 - u^2}}.
\]
and we can look this up: entry #28 in the table with $a = 1$ gives

$$\int \frac{dx}{\sqrt{x(1-x)}} = \int \frac{2\,du}{\sqrt{1-u^2}} = 2 \arcsin u + C = 2 \arcsin \sqrt{x} + C$$

as before.

3. Use entry #10 in the table with $b = 7$ and $a = 4$ (or the other way around is fine too). This gives

$$\int \sin 4x \sin 7x\,dx = \frac{1}{7^2 - 4^2} [4 \cos 4x \sin 7x - 7 \sin 4x \cos 7x] + C = \frac{1}{33} (4 \cos 4x \sin 7x - 7 \sin 4x \cos 7x) + C.$$

To do this by hand requires two integrations by parts, or a trig identity. We could take $u = \sin 4x$ and $dv = \sin 7x\,dx$, in which case $du = 4 \cos 4x\,dx$ and $v = -\frac{1}{7} \cos 7x$ and we have

(i) $$\int \sin 4x \sin 7x\,dx = -\frac{1}{7} \sin 4x \cos 7x + \frac{4}{7} \int \cos 4x \cos 7x\,dx.$$

Or we could take $u = \sin 7x$ and $dv = \sin 4x\,dx$, which will give the same thing except with all the 4’s and 7’s interchanged:

(ii) $$\int \sin 4x \sin 7x\,dx = -\frac{1}{4} \sin 7x \cos 4x + \frac{7}{4} \int \cos 4x \cos 7x\,dx.$$

If we multiply (i) by $\frac{7}{4}$ and (ii) by $\frac{4}{7}$ we have

(iii) $$\frac{7}{4} \int \sin 4x \sin 7x\,dx = -\frac{1}{4} \sin 4x \cos 7x + \int \cos 4x \cos 7x\,dx,$$

(iv) $$\frac{4}{7} \int \sin 4x \sin 7x\,dx = -\frac{1}{7} \sin 7x \cos 4x + \int \cos 4x \cos 7x\,dx.$$

Subtracting (iv) from (iii) gives

$$\left(\frac{7}{4} - \frac{4}{7}\right) \int \sin 4x \sin 7x\,dx = -\frac{1}{4} \sin 4x \cos 7x + \frac{1}{7} \sin 7x \cos 4x + C,$$

and, since $\frac{7}{4} - \frac{4}{7} = \frac{33}{28}$, if we multiply through by $\frac{28}{33}$ we get

$$\int \sin 4x \sin 7x\,dx = -\frac{7}{33} \sin 4x \cos 7x + \frac{4}{33} \sin 7x \cos 4x + C$$

as before. Another approach is to use the trig identities

$$\cos (7x - 4x) = \cos 7x \cos 4x + \sin 7x \sin 4x$$

$$\cos (7x + 4x) = \cos 7x \cos 4x - \sin 7x \sin 4x$$

Subtracting the second from the first we get $2 \sin 7x \sin 4x = \cos 3x - \cos 11x$. Therefore

$$\int \sin 4x \sin 7x\,dx = \frac{1}{2} \int (\cos 3x - \cos 11x)\,dx = \frac{1}{2} \left( \frac{\sin 3x}{3} - \frac{\sin 11x}{11} \right) + C.$$

The only problem with this is that it’s not obvious that it agrees with the previous answer. But it does, since

$$\frac{1}{2} \left( \frac{\sin 3x}{3} - \frac{\sin 11x}{11} \right) = \frac{1}{6} \sin (7x - 4x) - \frac{1}{22} \sin (7x + 4x)$$

$$= \frac{1}{6} [\sin 7x \cos 4x - \cos 7x \sin 4x] - \frac{1}{22} [\sin 7x \cos 4x + \cos 7x \sin 4x]$$

$$= \left(\frac{1}{6} - \frac{1}{22}\right) \sin 7x \cos 4x - \left(\frac{1}{6} + \frac{1}{22}\right) \cos 7x \sin 4x$$

$$= \frac{4}{33} \sin 7x \cos 4x - \frac{7}{33} \cos 7x \sin 4x.$$