1(a) We’ve done this one in class a couple of times, but it is potentially useful in 1(b), so I wanted you to go through it. The easiest method is to substitute $w = \sin \theta$, so that $dw = \cos \theta d\theta$ and we have

$$\int \sin \theta \cos \theta \, d\theta = \int w \, dw = \frac{w^2}{2} + C = \frac{1}{2} \sin^2 \theta + C.$$  

Or we could integrate by parts with $u = \sin \theta$ and $dv = \cos \theta \, d\theta$, in which case $v = \sin \theta = u$ and $du = \cos \theta \, d\theta = dv$. Then we have

$$\int \sin \theta \cos \theta \, d\theta = (\sin \theta)(\sin \theta) - \int \sin \theta \cos \theta \, d\theta,$$

so, adding the integral on the right to both sides,

$$2 \int \sin \theta \cos \theta \, d\theta = \sin^2 \theta + C,$$

which is equivalent to (1).

If you substituted for cosine instead of sine, or chose $u$ and $dv$ the opposite way when integrating by parts, you would get instead

$$\int \sin \theta \cos \theta \, d\theta = -\frac{1}{2} \cos^2 \theta + C.$$  

1(b) The simplest approach is to substitute $w = \sin \theta$ before integrating by parts. Then $dw = \cos \theta \, d\theta$ and we have

$$\int \ln (\sin \theta) \sin \theta \cos \theta \, d\theta = \int (\ln w) \, w \, dw.$$  

Now integrate this by parts with $u = \ln w$ and $dv = w \, dw$, so that $du = \frac{dw}{w}$ and $v = \frac{w^2}{2}$. This gives

$$\int (\ln w) \, w \, dw = \frac{w^2}{2} \ln w - \int \frac{w^2 \, dw}{w} = \frac{w^2}{2} \ln w - \frac{1}{2} \int w \, dw = \frac{w^2}{2} \ln w - \frac{1}{2} \left( \frac{w^2}{2} \right) + C = \frac{w^2}{2} \ln w - \frac{w^2}{4} + C.$$  

Since we substituted $w = \sin \theta$, this tells us that the original integral

$$\int \ln (\sin \theta) \sin \theta \cos \theta \, d\theta = \frac{1}{2} \sin^2 \theta \ln (\sin \theta) - \frac{1}{4} \sin^2 \theta + C.$$  

We can also do the integral without substituting by using (1). If we take $u = \ln (\sin \theta)$ and $dv = \sin \theta \cos \theta \, d\theta$, then (1) tells us that $v = \frac{1}{2} \sin^2 \theta$, and the chain rule tells us that $du = \frac{\cos \theta}{\sin \theta} \, d\theta$, and this implies

$$\int \ln (\sin \theta) \sin \theta \cos \theta \, d\theta = \left[ \ln (\sin \theta) \right] \left[ \frac{1}{2} \sin^2 \theta \right] - \int \left( \frac{1}{2} \sin^2 \theta \right) \frac{\cos \theta}{\sin \theta} \, d\theta.$$  

After simplifying the right side a little we have

$$\int \ln (\sin \theta) \sin \theta \cos \theta \, d\theta = \frac{1}{2} \sin^2 \theta \ln (\sin \theta) - \frac{1}{2} \int \sin \theta \cos \theta \, d\theta,$$
so referring to (1) again we finally obtain

\[
\int \ln (\sin \theta) \sin \theta \cos \theta \, d\theta = \frac{1}{2} \sin^2 \theta \ln (\sin \theta) - \frac{1}{4} \sin^2 \theta + C
\]
as before.

If we were using (2) instead of (1) then we would still have \( u = \ln (\sin \theta) \) and \( dv = \sin \theta \cos \theta \, d\theta \) and \( du = \frac{\cos \theta \, d\theta}{\sin \theta} \), but now we would have \( v = -\frac{1}{2} \cos^2 \theta \), and this gives

\[
\int \ln (\sin \theta) \sin \theta \cos \theta \, d\theta = [\ln (\sin \theta)] \left[ -\frac{1}{2} \cos^2 \theta \right] - \int \left( -\frac{1}{2} \cos^2 \theta \right) \frac{\cos \theta \, d\theta}{\sin \theta}
\]

(3)

To do this integral we can break off one cosine and use \( \cos^2 \theta = 1 - \sin^2 \theta \) on the other two, and then let \( w = \sin \theta \), so that \( dw = \cos \theta \, d\theta \). Then we have

\[
\int \frac{\cos^3 \theta \, d\theta}{\sin \theta} = \int \frac{1 - \sin^2 \theta}{\sin \theta} \cos \theta \, d\theta = \int \frac{1 - w^2}{w} \, dw
\]

\[
= \int (w^{-1} - w) \, dw = \ln |w| - \frac{w^2}{2} + C = \ln (\sin \theta) - \frac{1}{2} \sin^2 \theta + C.
\]

We didn’t need absolute values at the end since \( \sin \theta \) must be positive; otherwise the integral in 1(b) would be undefined. Combining this with (3) we finally obtain

\[
\int \ln (\sin \theta) \sin \theta \cos \theta \, d\theta = -\frac{1}{2} \cos^2 \theta \ln (\sin \theta) + \frac{1}{2} \left( \ln (\sin \theta) - \frac{1}{2} \sin^2 \theta \right) + C
\]

which is the same answer as before since \( 1 - \cos^2 \theta = \sin^2 \theta \).

2. This will require two integrations by parts to get rid of the two factors of \( x \). First take \( u = x^2 \) and \( dv = \sin 3x \, dx \), so that \( du = 2x \, dx \) and \( v = -\frac{1}{3} \cos 3x \). Then we have

(2) \[
\int x^2 \sin 3x \, dx = x^2 \left( -\frac{1}{3} \cos 3x \right) - \int \left( -\frac{1}{3} \cos 3x \right) 2x \, dx = -\frac{x^2}{3} \cos 3x + \frac{2}{3} \int x \cos 3x \, dx.
\]

In the remaining integral take \( u = x \) and \( dv = \cos 3x \, dx \), so that \( du = dx \) and \( v = \frac{1}{3} \sin 3x \). This gives

\[
\int x \cos 3x \, dx = x \left( \frac{1}{3} \sin 3x \right) - \int \left( \frac{1}{3} \sin 3x \right) \, dx
\]

\[
= \frac{x}{3} \sin 3x - \frac{1}{3} \left( -\frac{1}{3} \cos 3x \right) + C
\]

\[
= \frac{x}{3} \sin 3x - \frac{1}{9} \cos 3x + C.
\]

Combining this with (2) we finally get

\[
\int x^2 \sin 3x \, dx = -\frac{x^2}{3} \cos 3x + \frac{2}{3} \left( \frac{x}{9} \sin 3x + \frac{1}{9} \cos 3x \right) + C
\]

\[
= -\frac{x^2}{3} \cos 3x + \frac{2x}{9} \sin 3x + \frac{2}{27} \cos 3x + C.
\]