1. The simplest approach is to substitute \( w = x^6 \) before doing anything else. Then \( dw = 6x^5 \, dx \), so \( \frac{dw}{w} = x^6 \, dx \) and we have

\[
(1) \quad \int x^{11} \sin (x^6) \, dx = \int x^6 \sin (x^6) \, x^5 \, dx = \int w \sin w \, \frac{dw}{6} = \frac{1}{6} \int w \sin w \, dw.
\]

Now integrate by parts with \( u = w \) and \( dv = \sin w \, dw \), so that \( du = dw \) and \( v = -\cos w \) and we have

\[
\int w \sin w \, dw = w(-\cos w) - \int (-\cos w) \, dw = -w \cos w + \int \cos w \, dw = -w \cos w + \sin w + C.
\]

Combining this with (1), we conclude that

\[
\int x^{11} \sin (x^6) \, dx = \frac{1}{6} \int w \sin w \, dw = \frac{1}{6} (-w \cos w + \sin w) + C = \frac{1}{6} [-x^6 \cos (x^6) + \sin (x^6)] + C.
\]

The integral can be done without substituting in more or less the same way, but the substitution makes it clearer what to do. The only way that a power of \( x \) times \( \sin (x^6) \) will integrate nicely is if it integrates to a multiple of \( \cos (x^6) \), so let’s first calculate the derivative of \( \cos (x^6) \), which is \( -\sin (x^6) \cdot 6x^5 \); in other words, if we could take \( dv = -6x^5 \sin (x^6) \, dx \) then \( v \) would be \( \cos (x^6) \). This suggests that we should rewrite

\[
\int x^{11} \sin (x^6) \, dx = \int \frac{x^{11}}{-6x^5} [-6x^5 \sin (x^6)] \, dx = \int \left( -\frac{1}{6} x^6 \right) \left[ -6x^5 \sin (x^6) \right] \, dx
\]

and then take \( u = -\frac{1}{6} x^6 \), in which case \( du = -x^5 \, dx \). Then we have

\[
\int x^{11} \sin (x^6) \, dx = \left( -\frac{1}{6} x^6 \right) \cos (x^6) - \int \cos (x^6) (-x^5 \, dx) = \frac{1}{6} x^6 \cos (x^6) + \int x^5 \cos (x^6) \, dx.
\]

Now we could substitute \( w = x^6 \) again, or stay with the same thought process as before: the only way this integral could come out nicely is if it comes out to a multiple of \( \sin (x^6) \), so we calculate the derivative of \( \sin (x^6) \) and get \( \cos (x^6) \cdot 6x^5 \), which implies that

\[
\int x^{11} \sin (x^6) \, dx = -\frac{1}{6} x^6 \cos (x^6) + \int x^5 \cos (x^6) \, dx = -\frac{1}{6} x^6 \cos (x^6) + \frac{1}{6} \sin (x^6) + C
\]

as before.

2. In both integrals one wants to get rid of the powers of \( x \) by repeatedly integrating by parts. This means that \( \int x^2 e^{9x} \, dx \) will require two integrations by parts, whereas \( \int x^9 e^{2x} \, dx \) will take nine, so I hope everybody picked the first one. If we take \( u = x^2 \) and \( dv = e^{9x} \, dx \), then \( du = 2x \, dx \) and \( v = \frac{e^{9x}}{9} \) and we have

\[
(1) \quad \int x^2 e^{9x} \, dx = x^2 \left( \frac{e^{9x}}{9} \right) - \int \left( \frac{e^{9x}}{9} \right) 2x \, dx = \frac{x^2 e^{9x}}{9} - \frac{2}{9} \int x e^{9x} \, dx.
\]

In the remaining integral we again take \( dv = e^{9x} \, dx \), so that \( u = x \), \( du = dx \) and \( v = \frac{e^{9x}}{9} \) and we have

\[
\int x e^{9x} \, dx = x \left( \frac{e^{9x}}{9} \right) - \int \left( \frac{e^{9x}}{9} \right) dx = \frac{x e^{9x}}{9} - \frac{1}{9} \int e^{9x} \, dx = \frac{x e^{9x}}{9} - \frac{e^{9x}}{81} + C.
\]
Combining this with (1) we finally have

\[
\int x^2 e^{9x} \, dx = \frac{x^2 e^{9x}}{9} - \frac{2}{9} \left( \frac{x e^{9x}}{9} - \frac{e^{9x}}{81} \right) + C
\]

\[
= \frac{x^2 e^{9x}}{9} - \frac{2 x e^{9x}}{81} + \frac{2 e^{9x}}{729} + C.
\]

For the record,

\[
\int x^9 e^{2x} \, dx = \frac{e^{2x}}{8} \left( 4x^9 - 18x^8 + 72x^7 - 252x^6 + 756x^5 - 1890x^4 + 3780x^3 - 5670x^2 + 5670x - 2835 \right) + C.
\]

The best way of calculating this by hand is to use the idea in problem 48 in section 7.2: the answer has to be \( e^{2x} \) times some polynomial of degree 9, so write down \( e^{2x} \) times a polynomial of degree 9 with generic coefficients, calculate the derivative, and choose the coefficients to force the derivative to be \( x^9 e^{2x} \).