Fixed Point Theory
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Contents

Preface

These notes are intended as a brief introduction to topological fixed point theory presented as a mini-course at the Fourteenth Brazilian Topology Meeting held at UNICAMP in Campinas, July 26 - 30, 2004. The main objective is to present the basic ideas behind the theory beginning with the work of Solomon Lefschetz in the 1920s. We then address the converse of the Lefschetz fixed point theorem which brings up the notion of fixed point classes first introduced by Jakob Nielsen. Through the work of K. Reidemeister, W. Franz, and F. Wecken in the 1930s and early 1940s, Nielsen fixed point theory is given a solid foundation. We then discuss the computational aspect of the Nielsen number, in particular the Reidemeister trace. Applications to low dimensional dynamics will also be discussed.

Throughout some proofs are omitted or are only sketched. The basic references are [**1**], [**5**], [**8**], [**10**], and [**12**]. The survey article [**2**] gives a very nice historical account of the early development of the subject.

I thank the scientific and organizing committees for the kind invitation and the opportunity for me to give this course. I am grateful to Prof. Daniel Vendruscolo for translating the original manuscript to Portuguese.

Lecture I

1. Some Classical Theorems

Let K and K' be two finite simplicial complexes such that $|K| = |K'|$, i.e., same underlying topological space.

Define

$$\chi := \sum_q (-1)^q \#\{\ q - \text{simplicies in } K\};$$
$$\chi' := \sum_q (-1)^q \#\{\ q - \text{simplicies in } K'\}$$

As it turns out, these two integers are the same and we have the following

THEOREM 1.1.

$$\chi = \chi'.$$

We call $\chi = \chi(|K|)$ the Euler characteristic of $|K|$.

This result is an integrality type theorem, namely, the sum of a set of local variants turns out to be a global invariant. Euler characteristic is named after Leonard Euler who first realized the invariance of the alternating sum of the number of simplices for planar graphs.

More generally, if M is a compact polyhedron, we define

$$\begin{aligned}\chi(M) :&= \sum_q (-1)^q \mathrm{rank}_{\mathbb{Z}} H_q(M;\mathbb{Z}) \\ &= \sum_q (-1)^q b_q\end{aligned}$$

where b_q is the q-th Betti number of M.

One of the most beautiful results in geometry is the famous Gauss-Bonnet Theorem. In dimension two, it is stated as follows.

THEOREM 1.2. *Let $M \subset \mathbb{R}^3$ be a compact surface. Then*

$$2\pi \cdot \chi(M) = \int_M \kappa(p)$$

where $\kappa(p)$ denotes the Gaussian curvature at $p \in M$.

Clearly, the curvature is not a topological invariant but the theorem asserts that the totality of these local variants is indeed a topological invariant.

In complex analysis, one encounters contour integrals and therefore the winding number.

Let $\gamma : [0,1] \to \mathbb{R}^2 - \{0\}$ be a closed smooth curve. Define the *winding number* of γ to be

$$I_\gamma := \frac{1}{2\pi i} \int_\gamma \frac{dz}{z}.$$

THEOREM 1.3. *For any closed smooth curve c homotopic to γ, we have*

$$I_\gamma = I_c \quad (\in \mathbb{Z}).$$

Suppose M is a compact orientable smooth manifold and v is a smooth vectorfield on M with isolated (hence finite) zeroes. For the moment, assume that we are in some Euclidean space and 0 is an isolated zero of v. Then by taking a small ϵ-disk around 0, the boundary S_ϵ is homeomorphic to a sphere and for any $x \in S_\epsilon$, the map $x \mapsto v(x)/\|v(x)\|$ gives rise to an integer given by the degree. Using local parametrization in the general case, we have an integer-valued index i_x associated to each zero x. The well-known theorem of Poincaré-Hopf states that

THEOREM 1.4.

$$\chi(M) = \sum_x i_x$$

where x ranges over the set of zeroes of v.

Our first objective is to present the following celebrated theorem of Lefschetz-Hopf.

THEOREM 1.5. *Let X be a compact connected polyhedron and $f : X \to X$ a selfmap such that f has isolated fixed points each of which lies in the interior of a maximal simplex. Then*

$$L(f) = \sum_{x_i} I(f, x_i)$$

where $L(f)$ denotes the Lefschetz number and $I(f, x_i)$ is the fixed point index of f at the isolated fixed point x_i.

We shall see that $I(f, x_i)$ is defined in a small neighborhood U_i of x_i. In fact, if $I(f, x_i) \neq 0$ then any map (compactly fixed) homotopic to f (relative to $X - U_i$) must have a fixed point point in U_i.

Because of this theorem, $L(f) \neq 0$ implies that $Fixf := \{x \in X | f(x) = x\}$ is non-empty.

2. Brouwer Fixed Point Theorem

The Lefschetz-Hopf Fixed Point Theorem is not the first topological fixed point theorem of this type.

From Calculus, we have seen the following result, known as the Intermediate Value Theorem, due to B. Bolzano who proved this in 1817.

THEOREM 2.1. *Let $f : [-1, 1] \to \mathbb{R}$ be a continuous function such that $f(-1) < 0$ and $f(1) > 0$. Then there exists a point $c \in [-1, 1]$ such that $f(c) = 0$.*

The Intermediate Value Theorem (IVT) is a topological result about the real line. One can obtain the one dimensional Brouwer fixed point theorem from the IVT.

THEOREM 2.2. *Let $f : [-1, 1] \to [-1, 1]$ be a continuous function. There exists $c \in [-1, 1]$ such that $f(c) = c$.*

PROOF. Let $h(x) = x - f(x)$. The map h is continuous on $[-1, 1]$. If $f(1) = 1$ or $f(-1) = -1$ then we are done. Now, suppose $f(1) \neq 1$

and $f(-1) \neq -1$. Then, $h(-1) < 0$ and $h(1) > 0$. By the IVT, there exists $c \in [-1, 1]$ such that $h(c) = 0$, i.e., $f(c) = c$. $\square$

The following is the n-dimensional Brouwer fixed point theorem due to L.E.J. Brouwer (1912).

THEOREM 2.3. *Let D^n denote the closed n-disk in $\mathbb{R}^n$ and $f : D^n \to D^n$ a (continuous) map. Then there exists a point $x_0 \in D^n$ such that $f(x_0) = x_0$.*

In 1931, K. Borsuk observed that Theorem 2.3 is equivalent to the following *no-retraction* theorem.

THEOREM 2.4. *There is no continuous map $r : D^n \to \partial D^n$ with $r(x) = x$ for all $x \in \partial D^n$ where $\partial D^n = S^{n-1}$ is the boundary of D^n.*

Next, we will show that Theorem 2.3 is equivalent to Theorem 2.4.

Theorem 2.3 $\Rightarrow$ Theorem 2.4

PROOF. Suppose there exists an $r : D^n \to \partial D^n$ such that $r|_{\partial D^n} = \mathrm{id}_{\partial D^n}$. Let $i : \partial D^n \hookrightarrow D^n$ be the inclusion. Then,

$$(r \circ i)(x) = r(x) = x$$

and so the composite $(r \circ i)$ is the identity on ∂D^n.

Since $\partial D^n = S^{n-1}$ and $H_{n-1}(S^{n-1}) \cong \mathbb{Z}$,

$$\mathrm{id} = (r \circ i)_* : H_{n-1}(\partial D^n) \to H_{n-1}(\partial D^n)$$

is nonzero. However, the following diagram

$$\begin{tikzcd} H_{n-1}(\partial D^n) \arrow[rr, "(r\circ i)_*"] \arrow[dr, "i_*"'] & & H_{n-1}(\partial D^n) \\ & H_{n-1}(D^n) \arrow[ur, "r_*"'] & \end{tikzcd} \tag{2.1}$$

is commutative and $H_{n-1}(D^n) = 0$ and so $r_* \circ i_* = (r \circ i)_*$ must be the zero homomorphism which contradicts that $(r \circ i)_* = \mathrm{id}$. Hence, such an r cannot exist. $\square$

Theorem 2.4 $\Rightarrow$ Theorem 2.3

Proof. Suppose there exists a fixed point free map $f : D^n \to D^n$, i.e., a map such that $f(x) \neq x$ for all $x \in D^n$.

Define $g : D^n \to \partial D^n$ by sending x to the point of intersection between ∂D^n and the ray from x to $f(x)$. (See figure below.)

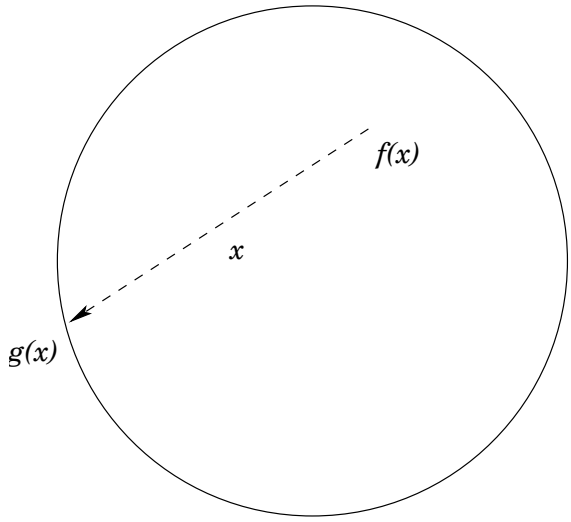

Figure 1

Now the map g is continuous and $g(x) = x$ for all $x \in \partial D^n$. But this contradicts the No-Retraction Theorem (2.4) and hence Theorem 2.3 holds. $\square$

The Lefschetz Fixed Point Theorem (or Theorem 1.5) generalizes the Brouwer Fixed Point Theorem since $L(f) = 1$ for any $f : D^n \to D^n$ for $H_q(D^n) = 0$ except when $q = 0$.

Next, we shall discuss the fixed point index.

3. The Fixed Point Index

Given a map $\varphi : S^n \to S^n$, the *degree* of φ, denoted by $\deg \varphi$, is the unique intger such that for any $x \in H_n(S^n) \cong \mathbb{Z}$,

$$\varphi_*(x) = \deg \varphi \cdot x.$$

Here, $\varphi_* : H_n(S^n) \to H_n(S^n)$ is the induced homomorphism in integral homology.

Suppose x_0 is an isolated fixed point of f. Choose a closed disk D_0^n centered at x_0 such that $D_0^n \cap Fixf = \{x_0\}$. For any $x \in \partial D_0^n$, $f(x) \neq x$ so we can define

$$\varphi(x) := \frac{x - f(x)}{\|x - f(x)\|}.$$

Thus, $\varphi : S^{n-1} \approx \partial D_0^n \to S^{n-1}$ is a continuous map. We then define the *fixed point index* $I(f, x_0)$ of f at x_0 to be the integer $\deg \varphi$.

Before we give the general definition of the fixed point index, let us recall the following fact. Given a pair (X, A), we have a long exact sequence of homology groups

$$... \to H_{n+1}(X, A) \to H_n(A) \to H_n(X) \to H_n(X, A) \to H_{n-1}(A) \to ... \quad .$$

In particular, if $X = D^n$ and $A = \partial X = S^{n-1}$ then for $q \neq 0$, $H_q(D^n) = 0$ and the long exact sequence yields $H_q(D^n, \partial D^n) \cong H_{q-1}(\partial D^n)$ for $q > 1$. Since $H_{q-1}(\partial D^n) = 0$ except for $q = n$, it follows that

$$H_n(D^n, \partial D^n) \cong H_{n-1}(\partial D^n) \cong \mathbb{Z}.$$

Note that the pair $(\mathbb{R}^n, \mathbb{R}^n - \{0\})$ has the same homotopy type as $(D^n, \partial D^n)$ and so $H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \cong \mathbb{Z}$.

3.1. Definition of $I(f, U)$ in $\mathbb{R}^n$. Since we are working with compact polyhedra which can be embedded in $\mathbb{R}^n$ for some n, we may assume that f is defined on some open subset of the Euclidean space.

Let $U \subset \mathbb{R}^n$ be an open subset and $f : U \to \mathbb{R}^n$ be a map. Assume that $Fixf$ is *compact in* U. Then the fixed point index of f is the integer $I(f, U) \in \mathbb{Z}$ such that

$$(i - f)_*(o_F) = I(f, U) \cdot o_0$$

where $i : U \hookrightarrow \mathbb{R}^n$, $(i - f)_* : H_n(U, U - Fixf) \to H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \cong \mathbb{Z}$, $o_F \in H_n(U, U - Fixf)$ is the fundamental homology class around $F = Fixf$ and $o_0 \in H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\})$ is the fundamental class around

the origin $\{0\}$. Note that if $F = Fixf = \{x_0\}$ consists of a single point, then $I(f, U) = \deg(i - f)$, in which case, we write $I(f, x_0) := I(f, U)$. For our purposes, $Fixf$ will be a finite set of (isolated) fixed points.

3.2. Basic Properties of $I(f, U)$. Let $f : U \to \mathbb{R}^n$ or $f : U(\subset X) \to X$ be a map where X is a polyhedron (assume that $Fixf$ is compact in U).

(1) If $I(f, U) \neq 0$ then f has a fixed point in U.

(2) (homotopy invariance) If $H : U \times [0, 1] \to X$ is a homotopy such that $\bigcup_t FixH_t \subset U$ is compact then

$$I(H_0, U) = I(H_1, U).$$

Remark 3.1. The compactness of $\bigcup_t FixH_t$ in U cannot be relaxed. For example, let U be the unit open disk in $X = \mathbb{R}^2$. Let $f : U \to X$ be the constant map at the origin $(0, 0)$. Define $H_t(x) = (t, 0), 0 \leq t \leq 1$. Note that $I(f, U) = 1$ but $FixH_1 = \emptyset$ so $I(H_1, U) = 0$. Here, $\bigcup_t FixH_t$ is not compact in U. (See figure below.)

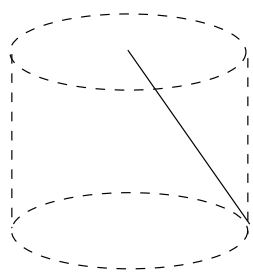

Figure 2

(3) (additivity) Suppose $U_1, ..., U_r$ are mutually disjoint open subsets of U and $Fixf \subset \bigcup U_j$. Let $f_j = f|_{U_j}$. Then,

$$I(f, U) = \sum_j I(f_j, U_j).$$

(4) (multiplicativity) Given $f : U \to X, g : V \to Y$, consider the product map

$$f \times g : U \times V (\subset X \times Y) \to X \times Y.$$

Then,

$$I(f \times g, U \times V) = I(f, U) \cdot I(g, V).$$

(5) (commutativity) Let U, V be open subsets of X, Y respectively. Given two maps $f : U \to Y$ and $g : V \to X$. Consider the maps

$$g \circ f : f^{-1}(V) \to X, \qquad f \circ g : g^{-1}(U) \to Y.$$

Then, $Fix(g \circ f)$ is homeomorphic to $Fix(f \circ g)$ and

$$I(g \circ f, f^{-1}(V)) = I(f \circ g, g^{-1}(U)).$$

We say that a map $f : U$ (open subset of X) $\to X$ is *compactly fixed* if $Fixf$ is compact in U. A homotopy $\{H_t\} : U \to X$ is *compactly fixed* if $(\bigcup_t FixH_t)$ is compact in U.

(6) (local removability) Suppose f has an isolated fixed point x_0 such that $I(f, U_0) = 0$ for some open neighborhood U_0 of x_0 such that $Fixf \cap U_0 = \{x_0\}$. For any open neighborhood V of x_0 with $V \subset U_0$, there exists a map $g : U \to X$ compactly fixed homotopic to f such that $g \equiv f$ on $U - V$ and $Fixg = Fixf - \{x_0\}$.

Note that from the additivity property of the fixed point index, if W is an open set such that $Fixf \subset W \subset cl(W) \subset U$ then $I(f, U) = I(f|_W, W)$.

Definition 3.1. Let $f : X \to X$ be a selfmap of a compact connected polyhedron X. For each $q \geq 0$, the rational homology $H_q(X; \mathbb{Q})$ is a finite dimensional vectorspace over $\mathbb{Q}$ and the induced homomorphism $f_{*q} : H_q(X; \mathbb{Q}) \to H_q(X; \mathbb{Q})$ is a linear transformation and thus the trace $\mathrm{tr} f_{*q}$ is well-defined. The Lefschetz number $L(f)$ of f is defined to be the number

$$L(f) := \sum_{q=0} (-1)^q \mathrm{tr} f_{*q}.$$

Note that it does *not* follow from the definition that $L(f)$ is an integer but a consequence of the Lefschetz-Hopf Theorem will imply that indeed $L(f) \in \mathbb{Z}$.

The next property is the same as the Lefschetz-Hopf Theorem.

(7) (normalization)

$$L(f) = I(f, X).$$

In particular, if f has isolated fixed points $x_1, ..., x_k$ then

$$L(f) = \sum_j I(f, x_j).$$

Remark 3.2. Properties **(1) - (5)** and **(7)** characterize the Lefschetz number. That is, any integer-valued function satisfying these properties coincides with the fixed point index.

There is a recent proof of the Lefschetz-Hopf Theorem due to M. Arkowitz and R.F. Brown, "The Lefschetz-Hopf theorem and axioms for the Lefschetz number," *Fixed Point Theory and Applications*, **1** (2004), 1 - 11.

Let X be a topological space of the homotopy type of a finite connected CW-complex. Let λ be an integer-valued function such that

(1) (homotopy) If $f, g : X \to X$ are maps and f is homotopic to g, then $\lambda(f) = \lambda(g)$.

(2) (cofibration) If $A \subset X$ is a subpolyhedron and given the following commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & X & \longrightarrow & X/A \\ {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle \bar{f}}\downarrow \\ A & \longrightarrow & X & \longrightarrow & X/A \end{array}$$

then

$$\lambda(f) = \lambda(f') + \lambda(\bar{f}).$$

(3) (commutativity) If $f : X \to Y, g : Y \to X$ are maps then

$$\lambda(f \circ g) = \lambda(g \circ f).$$

(4) (wedge of circles) If

$$f : \bigvee_{j=1}^{k} S_j^1 \to \bigvee_{j=1}^{k} S_j^1$$

is a map, $k \geq 1$ then

$$\lambda(f) = -(\deg f_1 + ... \deg f_k)$$

where $f_i = p \circ f \circ e_i, e_i : S_i^1 \hookrightarrow \bigvee_j^k S_j^1$ and $p_i : \bigvee_j^k S_j^1 \to S_i^1$ is the canonical projection.

The following is due to Arkowitz and Brown.

THEOREM 3.1. *If λ satisfies the axioms* **(1) - (4)** *then $\lambda = \tilde{L}$ where $\tilde{L}(f) := L(f) - 1$ is the reduced Lefschetz number.*

Now, by showing that $\tilde{I} := I - 1$ satisfies axioms **(1) - (4)**, the normalization property holds. Here, $\tilde{I}(f, X) := I(f, X) - 1$ is the reduced fixed point index.

4. Lefschetz's idea

How do we prove the Lefschetz-Hopf Fixed Point Theorem? S. Lefschetz first announced (1923) his fixed point theorem for selfmaps of compact manifolds. How did he come up with *his number*?

Let M be a compact connected orientable smooth manifold and let Δ_M denote the diagonal of M in $M \times M$, i.e., $\Delta_M := \{(x, x) | x \in M\} \subset M \times M$. Given a selfmap $f : M \to M$, we consider the map $1 \times f : M \to M \times M$ given by $(1 \times f)(x) = (x, f(x))$.

The image of $1 \times f$ is simply the graph of f.

We may assume, without loss of generality, that f is smooth and the graph of f and Δ_M are transverse to each other. Note that at an intersection (x, x), the point x is a fixed point of f. That is, fixed points of f are the intersections of the graph of f and the diagonal Δ_M.

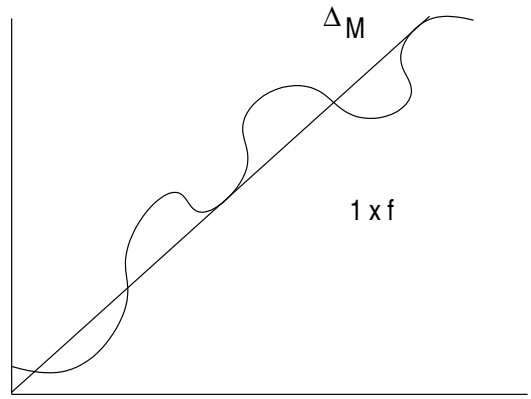

FIGURE 3

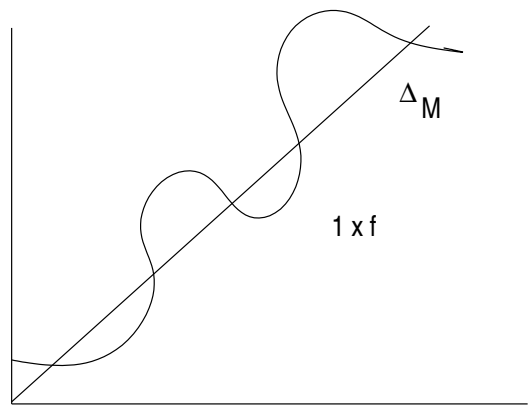

FIGURE 4

Each (transverse) intersection (x, x) can be assigned an intersection number $L_x(f) \in \{\pm 1\}$ (also known as the local Lefschetz number of f at x). It turns out that the Lefschetz number is simply the sum of the local Lefschetz numbers, i.e.,

$$L(f) = \sum_{x \in Fix f} L_x(f).$$

As we shall see, the definition of the Lefschetz number captures this geometric interpretation of fixed points as intersections.

Lecture II

5. Hopf Trace Theorem

Let X be a compact polyhedron. Since X is compact, there are only finitely many simplices. Suppose x_0 is an isolated fixed point of a selfmap f lying in the interior of a maximal simplex σ. By simplicial approximation, we may assume that f is simplicial so that f maps a simplex to a simplex. Since $f(x_0) = x_0$ and σ is maximal, we have $f(\sigma) \subset \sigma$. On the other hand, if σ' is a simplex such that $f(\sigma') \subset \sigma'$, then by the Brouwer fixed point theorem 2.3, f has a fixed point in σ'.

Denote by $C_p(X;\mathbb{Q})$ be the finite dimensional $\mathbb{Q}$-vectorspace generated by the oriented p-simplices of X. The map f induces

$$f_{\#p} : C_p(X;\mathbb{Q}) \to C_p(X;\mathbb{Q})$$

which is a chain map of the chain complex $\{C_p(X;\mathbb{Q}), \partial\}$ where ∂ is the usual boundary operator.

If $\sigma_1^p, ..., \sigma_{k(p)}^p$ are the oriented p-simplices, they generate $C_p(X;\mathbb{Q})$ so that the induced map $f_{\#p}$ is given by a matrix (a_{ij}) where

$$f_{\#p}(\sigma_j^p) = \sum_{i=1}^{k(p)} a_{ij}\sigma_i^p.$$

In particular, if $a_{ii} \neq 0$ for some i then f has a fixed point in σ_i^p.

We define

$$\mathcal{T}r(f_\#) := \sum_{q=0} (-1)^q \mathrm{tr}(f_{\#q}). \tag{5.1}$$

Recall from Definition 3.1 that the Lefschetz number is defined in a similar way at the homology level, that is,

$$L(f) = \mathcal{T}r(f_*) = \sum_{q=0} (-1)^q \mathrm{tr}(f_{*q}).$$

Before we state and prove the Hopf Trace Theorem, we need the following result which can be proven using linear algebra.

LEMMA 5.1. *Let A, B and C be finite dimensional vectorspaces over a field k such that*

$$0 \to A \to B \to C \to 0$$

is exact. Given a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \varphi_A \big\downarrow & & \varphi_B \big\downarrow & & \varphi_C \big\downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

we have

$$\mathrm{tr}\varphi_B = \mathrm{tr}\varphi_A + \mathrm{tr}\varphi_C.$$

Here is the Hopf Trace Theorem.

THEOREM 5.1.

$$\mathcal{T}r(f_\#) = \mathcal{T}r(f_*) = L(f).$$

PROOF. Let Z_i and B_i denote the i-th cycles and the i-th boundaries in $C_i = C_i(X; \mathbb{Q})$. Using the boundary map ∂_i,

$$0 \longrightarrow Z_i \hookrightarrow C_i \xrightarrow{\partial_i} B_{i-1} \longrightarrow 0$$

is exact.

Since $f_\#$ is a chain map, we have the following commutative diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z_i & \longrightarrow & C_i & \longrightarrow & B_{i-1} & \longrightarrow & 0 \\ & & f_{\#i}|_{Z_i} \big\downarrow & & f_{\#i} \big\downarrow & & f_{\#i-1}|_{B_{i-1}} \big\downarrow & & \\ 0 & \longrightarrow & Z_i & \longrightarrow & C_i & \longrightarrow & B_{i-1} & \longrightarrow & 0 \end{array}$$

Now Lemma 5.1 impies that

$$\mathrm{tr}(f_{\#i}) = \mathrm{tr}(f_{\#i}|_{Z_i}) + \mathrm{tr}(f_{\#i-1}|_{B_{i-1}}).$$

By the definition of homology, $H_i = Z_i/B_i$ and so the following is commutative.

$$\begin{tikzcd}
0 \arrow[r] & B_i \arrow[r] \arrow[d, "f_{\#i}|_{B_i}"'] & Z_i \arrow[r] \arrow[d, "f_{\#i}"'] & H_i \arrow[r] \arrow[d, "f_{*i}"'] & 0 \\
0 \arrow[r] & B_i \arrow[r] & Z_i \arrow[r] & H_i \arrow[r] & 0
\end{tikzcd}$$

By Lemma 5.1, we have

$$\mathrm{tr}(f_{*i}) = \mathrm{tr}(f_{\#i}|_{Z_i}) - \mathrm{tr}(f_{\#i}|_{B_i}).$$

Suppose that $\dim X = n$. Then, on one hand,

$$\begin{aligned}
\mathcal{T}r(f_{\#}) &= \sum_{q=0}^{n} (-1)^q \mathrm{tr}(f_{\#q}) \\
&= \sum_{q=0}^{n} [(-1)^q \mathrm{tr}(f_{\#q}|_{Z_q}) + (-1)^q \mathrm{tr}(f_{\#q-1}|_{B_{q-1}})].
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\mathcal{T}r(f_{*}) &= \sum_{q=0}^{n} (-1)^q \mathrm{tr}(f_{*q}) \\
&= \sum_{q=0}^{n} [(-1)^q \mathrm{tr}(f_{\#q}|_{Z_q}) + (-1)^{q+1} \mathrm{tr}(f_{\#q}|_{B_q})].
\end{aligned}$$

Since $B_{-1} = 0$, we have

$$\mathcal{T}r(f_{\#}) = \sum_{q=0}^{n} (-1)^q \mathrm{tr}(f_{\#q}|_{Z_q}) + \sum_{q=0}^{n-1} (-1)^{q+1} \mathrm{tr}(f_{\#q}|_{B_q}).$$

Simialrly, $B_{n+1} = 0$, so

$$\begin{aligned}
\mathcal{T}r(f_{*}) &= \sum_{q=0}^{n} (-1)^q \mathrm{tr}(f_{\#q}|_{Z_q}) + \sum_{q=0}^{n-1} (-1)^{q+1} \mathrm{tr}(f_{\#q}|_{B_q}) \\
&= \mathcal{T}r(f_{\#}).
\end{aligned}$$

□

Now, as an immediate consequence of the Hopf Trace Theorem, we have

THEOREM 5.2. *Let $f : X \to X$ be a selfmap of a compact connected polyhedron. If $L(f) \neq 0$ then f has a fixed point.*

6. Lefschtez-Hopf Fixed Point Theorem

The Lefschetz-Hopf Fixed Point Theorem asserts more than Theorem 5.2.

THEOREM 6.1. *Let X be a compact connected polyhedrfon. Suppose that $f : X \to X$ is a map with finitely many fixed points each of which lies in the interior of a maximal simplex. Then*

$$L(f) = \sum_{x_i \in Fixf} I(f, x_i).$$

In particular, $L(f) \in \mathbb{Z}$.

PROOF. If $x_0 \in |\sigma|$ is an isolated fixed point in the interior of a maximal simplex σ, since f may be assumed to be simplicial, we have $f(|\sigma|) \subset |\sigma|$. Let $p = \dim \sigma$. The contribution of x_0 to the trace $f_{\#p}$ is simply the fixed point index of f at x_0. □

The fact that the Lefschetz number is always an integer is still not clear unless f satisfies the hypothesis of the theorem. The following result, also known as Hopf's construction (see [**1**]), provides exactly this.

THEOREM 6.2. *Let $f : X \to X$ be a map on a compact connected polyhedron X. For any $\epsilon > 0$, there exists a map $f' : X \to X$ such that (1) $d(f(x), f'(x)) < \epsilon$ for all $x \in X$ where d is the metric on X; (2) f' is homotopic to f; and (3) $\#Fixf' < \infty$ and each fixed point of f' lies in the interior of a maximal simplex.*

By definition, the Lefschetz number is homological in nature, and thus is invariant under homotopy. That is, if f is homotopic to f' then $L(f) = L(f')$.

Let's see how one computes the Lefschetz number. As an easy example, we take selfmaps of a sphere.

EXAMPLE 6.1. Let $f : S^n \to S^n$ be a selfmap where S^n is the unit n-sphere with $n \geq 1$. If we use cellular homology then we can decompose $S^n = \{e_0\} \cup \{\sigma^n\}$ as a union of two cells, one in dimension 0 and one in dimension n. Note that

$$H_i(S^n) = \begin{cases} 0, & \text{if } i \neq 0, n; \\ \mathbb{Z}, & \text{otherwise.} \end{cases}$$

It follows that $f_{*0} = \mathrm{id}$ and thus $\mathrm{tr}(f_{*0}) = 1$. The homomorphism $f_{*n} : \mathbb{Z} \to \mathbb{Z}$ is a 1×1 matrix so $\mathrm{tr}(f_{*n}) = \deg f$. Hence,

$$L(f) = 1 + (-1)^n \deg f.$$

7. Borsuk-Ulam Theorem

In this section, we make use of the Lefschetz number to give a short proof of the famous (classical) Borsuk-Ulam Theorem.

THEOREM 7.1. *Let $f : S^2 \to \mathbb{R}^2$ be a map. Then exists a point $z \in S^2$ such that $f(z) = f(-z)$ where $-z$ denotes the antipodal point of z.*

If we write $f(z) = (h_1(z), h_2(z))$ then we can interpret Theorem 7.1 as follows:

At any given time, there is a location z on earth (S^2) whose temperature ($h_1(z)$) and barometric pressure ($h_2(z)$) are the same as those at its antipodal point $-z$.

Our proof will make use of the notion of group actions on topological spaces.

7.1. G-spaces and G-maps. Let G be a finite group. A *G-action* on a topological space X is a continuous map

$$\Phi : G \times X \to X$$

such that (i) $\Phi(e, x) = x$ for all $x \in X$ where $e \in G$ denotes the unity; (ii) if we write $gx := \Phi(g, x)$ then for any $g_1, g_2 \in G$ and for any $x \in X$,

$$g_1(g_2 x) = (g_1 g_2)(x)$$

or

$$\Phi(g_1, \Phi(g_2, x)) = \Phi(g_1 g_2, x).$$

In this case, X is a called a *G-space*.

A map $\varphi : X \to Y$ between two G-spaces is said to be a *G-map* (or G-equivariant map) if for any $g \in G$ and any $x \in X$,

$$\varphi(gx) = g\varphi(x).$$

Equivalently, if we let $\Phi_1 : G \times X \to X, \Phi_2 : G \times Y \to Y$ be the respective actions then

$$\varphi(\Phi_1(g, x)) = \Phi_2(g, \varphi(x)).$$

Now we return to the Borsuk-Ulam setting.

Let $G = \mathbb{Z}_2$ act on S^2 and $\mathbb{R}^2$ via the antipodal action. In other words, G consists of two elements - one is the identity map and the other is the antipodal map.

For any $f : S^2 \to \mathbb{R}^2$, define φ by

$$\varphi(z) = f(z) - f(-z).$$

Write $G = \mathbb{Z}_2 = \{\pm 1\}$. Note that

$$\varphi(-z) = f(-z) - f(z) = -\varphi(z)$$

so that φ is a $\mathbb{Z}_2$-map.

Using the language of G-spaces and G-maps, we can reformulate the classical Borsuk-Ulam Theorem. The following is equivalent to Theorem 7.1.

THEOREM 7.2. *For any $\mathbb{Z}_2$-map $\varphi : S^2 \to \mathbb{R}^2$ (with the antipodal actions on both S^2 and $\mathbb{R}^2$), there exists a point $z \in S^2$ such that $\varphi(z) = 0$. Equivalently, there are no $\mathbb{Z}_2$-maps from S^2 to $\mathbb{R}^2 - \{0\}$.*

PROOF. Suppose $\varphi^{-1}(\{0\}) = \emptyset$. Then φ is a map from S^2 to $\mathbb{R}^2 - \{0\}$.

Note that $\mathbb{R}^2 - \{0\}$ has the same $\mathbb{Z}_2$ homotopy type as the unit circle S^1. In fact, S^1 is a $\mathbb{Z}_2$-invariant strong deformation retract of $\mathbb{R}^2 - \{0\}$. To see that, we find a $\mathbb{Z}_2$-retraction from $\mathbb{R}^2 - \{0\}$ to S^1,

sending z to the point on S^1 radially. More precisely, if z is inside the unit circle, z is sent to S^1 following the ray from the origin to z. If z is outside the unit circle, z is sent to S^1 by retracting along the ray from z to the origin. (See figure below.)

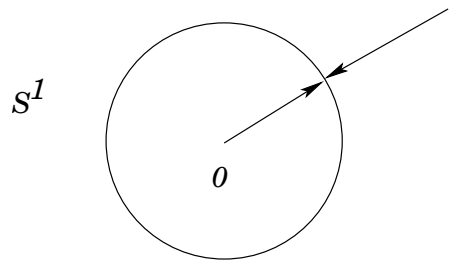

FIGURE 5

Since S^1 with the antipodal action is a G-subspace of S^2, we consider the composite map

$$\psi : S^2 \xrightarrow{\varphi} \mathbb{R}^2 - \{0\} \sim_{\mathbb{Z}_2} S^1 \hookrightarrow S^2.$$

As a selfmap of S^2, the Lefschetz number of ψ is given by

$$L(\psi) = 1 + \deg \psi.$$

However, ψ factors through S^1 and $H_2(S^1) = 0$, it follows that $\deg \psi = 0$ and so $L(\psi) = 1$.

A careful examination of Hopf's construction Theorem 6.2 yields an equivariant analog, i.e., every G-map can be deformed equivariantly to a G-map with finitely many fixed points. Note that if z_0 is a fixed point of ψ then $\psi(-z_0) = -\psi(z_0) = -z_0$ so $-z_0$ is also a fixed point of ψ. Furthermore, take a euclidean neighborhood U_0 of z_0 containing no other fixed points. (See figure below.)

Since ψ is a $\mathbb{Z}_2$-map, $-(i - \psi)(z) = \psi(z) - z = (i - \psi)(-z)$. The antipodal map is a homeomorphism and it has non-zero degree. It follows from the commutativity property of the fixed point index that

$$I(\psi, z_0) = I(\psi, -z_0).$$

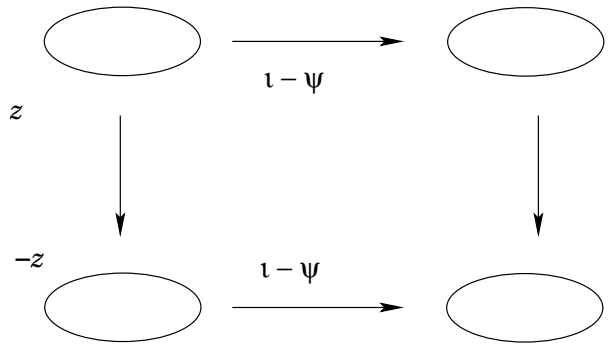

FIGURE 6

Now the fixed point set of ψ looks like $Fix\psi = \{\pm z_0, \pm z_1, ..., \pm z_k\}$ and the normalization property asserts that

$$L(\psi) = \sum_i I(\psi, \pm z_i) = 2\sum_i I(\psi, z_i).$$

Hence, $L(\psi)$ must be divisible by 2. Clearly 2 does not divide 1 and we have a contradiction. □

REMARK 7.1. In fact, if G is a finite group acting freely and simplicially on a finite simplicial complex X, then for any $f : X \to X$, $|G|$ divides $L(f)$. Here, by a free action, we mean that for every $x \in X$, $G_x := \{g \in G | gx = x\}$ is the trivial subgroup.

8. A Simple Example

Let X be the 'figure-eight'. (See figure below.)

The fundamental group of X with basepoint x_0 is

$$\pi_1(X, x_0) \cong \langle \alpha, \beta \rangle \cong \mathbb{Z} * \mathbb{Z}.$$

It is isomorphic to the free group on two generators. On the other hand, in dimension one, the integral homology

$$H_1(X) \cong \mathbb{Z} \oplus \mathbb{Z} \cong \langle \bar{\alpha} \rangle \oplus \langle \bar{\beta} \rangle$$

where $\bar{\alpha}$ is the image of α in $H_1 = \pi_1/[\pi_1, \pi_1]$.

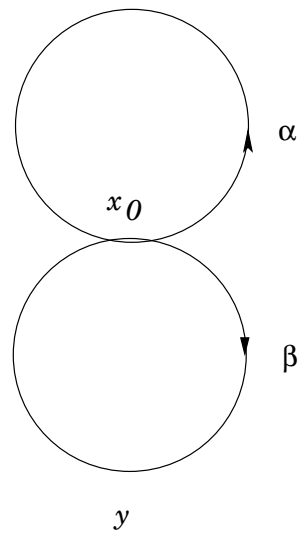

FIGURE 7

Let $f : X \to X$ be a map such that $f_\#(\alpha) = \alpha^2$ and $f_\#(\beta) = \beta^{-1}$ so that f has two fixed points x_0 and y.

Now, $\mathrm{tr}(f_{*0}) = 1$ and

$$f_{*1} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$$

since $f_{*1}(\bar{\alpha}) = 2\bar{\alpha}$ and $f_{*1}(\bar{\beta}) = -\bar{\beta}$.

It follows that

$$L(f) = 1 - \mathrm{tr}(f_{*1}) = 1 - (2 - 1) = 0.$$

As we shall see that the Nielsen number $N(f)$ is equal to 2 and so for any map f' homotopic f, $Fixf' \neq \emptyset$. More precisely, the fixed points x_0 and y actually belong to two different *fixed point classes*. One has index $I(f, y) = +1$ and the other has index $I(f, x_0) = -1$.

This example shows that the converse Lefschetz theorem does not hold in general, i.e., $L(f) = 0$ is not sufficient to removing all fixed points under homotopy. The key concept is that of Nielsen fixed point theory.

Given a map $f : X \to X$ such that $Fixf \neq \emptyset$. Two fixed points $x, y \in Fixf$ are *Nielsen equivalent* (as fixed points of f) if there exists a path $C : [0,1] \to X, C(0) = x, C(1) = y$ such that $f \circ C \sim C$ relative to endpoints. (See figure below.)

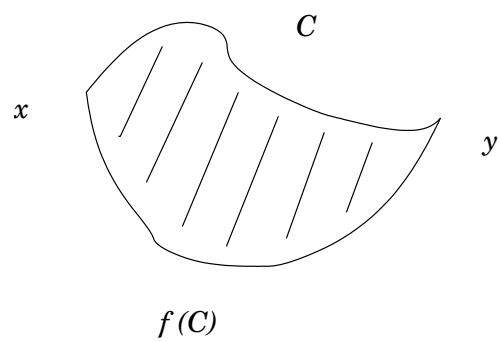

FIGURE 8

This gives rise to an equivalence relation on $Fixf$ and the equivalence classes are called *Nielsen (fixed point) classes*. Since $Fixf$ is compact, there are only finite number of such classes. Given a fixed point class $\mathcal{F}$, the fixed point index $I(f, \mathcal{F}) := I(f, U)$ is well-defined where U is a neighborhood of $\mathcal{F}$ such that $U \cap Fixf = \mathcal{F}$. The class $\mathcal{F}$ is called *essential* if $I(f, \mathcal{F}) \neq 0$, otherwise, it is called *inessential*.

DEFINITION 8.1. The Nielsen number of f is defined by

$$N(f) := \#\{\mathcal{F} | I(f, \mathcal{F}) \neq 0\}.$$

In other words, the Nielsen number is simply the number of essential Nielsen classes.

Let's return to the 'figure-eight' example and we will show that x_0 and y belong to distinct Nielsen classes. To see that, let u be the right half circle from x_0 to y. Suppose w is a path from x_0 to y such that $w \sim f \circ w$. Let $\gamma = [wu^{-1}] \in \pi_1(X, x_0)$. Then

$$\gamma \beta f_{\#}(\gamma)^{-1} = [wu^{-1}]\beta[f \circ u][f \circ w^{-1}] = 1.$$

In H_1, $\gamma\beta f_{\#}(\gamma)^{-1}$ projects to $\bar{\gamma}+\bar{\beta}-f_{*1}(\bar{\gamma})$. Suppose $\bar{\gamma}=a\bar{\alpha}+b\bar{\beta}$ for some $a,b\in\mathbb{Z}$. We have

$$\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -a \\ 2b+1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

But $2b+1\neq 0$ since $b\in\mathbb{Z}$. Hence, $\gamma\beta f_{\#}(\gamma)^{-1}$ cannot be homotopic to the trivial loop or equivalently, x_0 and y are not Nielsen equivalent as fixed points of f.

Lecture III

9. Nielsen Fixed Point Theory

From the simple example in the last lecture, we have seen that the notion of fixed point classes may be used to explain subtle behavior that the Lefschetz theory cannot detect. Recall that the Nielsen number of a map is simply the number of essential Nielsen classes. In this section, we will discuss some basic properties of the Nielsen number.

(1) $0 \leq N(f) \leq \#Fixf$.

By definition, $N(f)$ is a non-negative integer. Since each Nielsen class contains at least one fixed point of f, it follows that $N(f) \leq \#Fixf$.

(2) If $f' \sim f$ then $N(f') = N(f)$.

To see that the Nielsen number is a homotopy invariant, let $\{f_t\}$ be a homotopy so that $f_0 = f$ and $f_1 = f'$. Consider the map $F : X \times [0,1] \to X \times [0,1]$ given by $F(x,t) = (f_t(x), t)$.

Note that $(x,t) \in FixF$ iff $x \in Fixf_t$. Let $\mathcal{F}$ be the fixed point class of F containing $(x_0, 0)$. Let $\mathcal{F}_0 = \mathcal{F} \cap (X \times \{0\})$ and $\mathcal{F}_1 = \mathcal{F} \cap (X \times \{1\})$. Furthermore, write $\mathbb{F}_0 = pr_1(\mathcal{F}_0)$ and $\mathbb{F}_1 = pr_1(\mathcal{F}_1)$ where pr_1 is the projection onto X.

Observe that $\mathbb{F}_0$ is a fixed point class of f_0 and $\mathbb{F}_1$ is a fixed point class of f_1 (possibly empty).

By the homotopy invariance of the fixed point index, we have

$$I(f_0, \mathbb{F}_0) = I(f_1, \mathbb{F}_1).$$

This implies that $N(f)$ is a homotopy invariant. Unlike the Lefschetz number $L(f)$ whose non-vanishing implies existence of fixed points, $N(f)$ gives possible multiple fixed points provided we can compute

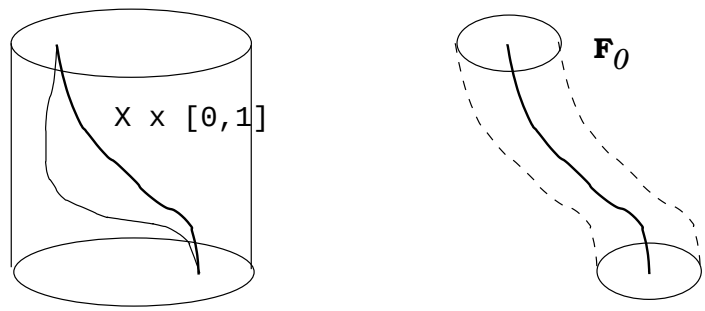

FIGURE 9

$N(f)$. Originally, J. Nielsen employed this notion of fixed point classes for studying surface automorphisms.

The following classical result of F. Wecken makes the Nielsen number an important object of study.

THEOREM 9.1. *Suppose X is a compact connected triangulated manifold of dimension at least 3. For any $f : X \to X$, there exists a map f' homotopic to f such that $\#Fixf' = N(f)$.*

PROOF. (Sketch) (1) Deform f to f_1 such that $\#Fixf_1 < \infty$.

(2) If $x \in Fixf_1$ has index 0 then we can remove x locally, i.e., there exists $f_2 \sim f_1$ such that for any $y \in Fixf_2, I(f_2, y) \neq 0$.

(3) Suppose x and y belong to the same Nielsen class, i.e., there exists a path $C : [0,1] \to X, C(0) = x, C(1) = y$ such that $f \circ C \sim C$. Choose a contractible neighborhood U of C (may assume that U does not contain any other fixed points). (See figure below.)

Deform f_2 to f_2' relative to $X - U$ such that $Fixf_2' = Fixf_2 - \{y\}$. (Note that the local index of f_2' at x is different from that of f_2 at the same point.) Repeating this process a finite number of times, we will arrive at a map $f' \sim f$ such that each Nielsen class of f' has exactly one fixed point and has non-zero index. □

In practice, we must use algebraic techniques to compute $N(f)$. Next we present Nielsen fixed point theory from the covering space approach.

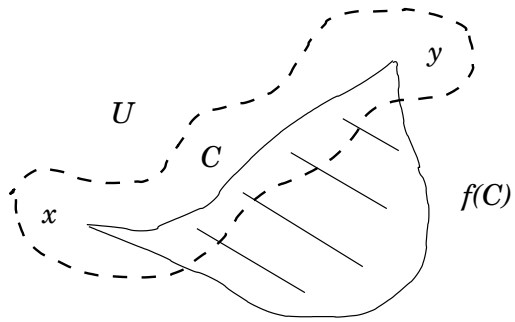

FIGURE 10

10. Covering Space Approach

Let X be a compact connected polyhedron. Denote by $\eta : \tilde{X} \to X$ the universal cover of X. Suppose $f : X \to X$ is a map. Let $\tilde{f} : \tilde{X} \to \tilde{X}$ be a lift of f to the universal cover so that

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{X} \\ {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle\eta} \\ X & \xrightarrow[f]{} & X \end{array}$$

is commutative.

The group of deck transformations of η is simply the set of all lifts of the identity map 1_X. We denote this set by $Cov\eta$ (covering group).

Choose a basepoint $\tilde{x}_0 \in \tilde{X}$ and let $\alpha \in Cov\eta$. By the uniqueness of lifts, there exists a unique element $\tilde{\alpha} \in Cov\eta$ such that

$$\tilde{\alpha}(\tilde{f}(\tilde{x}_0)) = \tilde{f}(\alpha(\tilde{x}_0)).$$

Let the assignment $\alpha \mapsto \tilde{\alpha}$ be $\varphi : Cov\eta \to Cov\eta$. It follows that

$$\tilde{f}\alpha = \varphi(\alpha)\tilde{f}, \qquad \text{for all } \alpha \in Cov\eta.$$

If $\beta \in Cov\eta$, then

$$\varphi(\alpha)\varphi(\beta)\tilde{f} = \varphi(\alpha)\tilde{f}\beta = (\tilde{f}\alpha)\beta = \tilde{f}(\alpha\beta) = \varphi(\alpha\beta)\tilde{f}$$

which implies that $\varphi(\alpha\beta) = \varphi(\alpha)\varphi(\beta)$ so φ is a group homomorphism.

Note that if $\tilde{f}(\tilde{x}) = \tilde{x}$ then $\eta\tilde{f}(\tilde{x}) = \eta\tilde{x}$ and so $\eta\tilde{x} = f\eta(\tilde{x})$. In other words, $\eta(\tilde{x}) \in Fixf$. Thus, fixed points of lifts of f project to fixed points of f.

Example 10.1. Take $X = S^1$ and so $\tilde{X} = \mathbb{R}$. Let $f : S^1 \to S^1$ be defined by

$$f(e^{it}) = e^{i(\pi - t)}.$$

Suppose we let $\tilde{f}_k(t) = -t + k\pi$ where $k \in \mathbb{Z}$. Then $\tilde{f}_k$ is a lift of f iff $\eta\tilde{f}_k(t) = f\eta(t) = e^{i(\pi-t)}$. This in turn is equivalent to $\eta(-t + k\pi) = e^{-it}e^{ik\pi} = e^{i(\pi-t)}$ which is the same as saying that k must be odd. When $k = 1$, $Fix\tilde{f}_1 = \{\frac{\pi}{2}\}$. For $k = 5$, $Fix\tilde{f}_5 = \{\frac{5\pi}{2}\}$. However,

$$\eta Fix\tilde{f}_1 = \{e^{i\pi/2}\} = \eta Fix\tilde{f}_5$$

so different lifts may yield the same fixed points of f.

In general, if $\tilde{f}$ is a lift and $\gamma \in Cov\eta$ then

$$\eta Fix\tilde{f} = \eta Fix(\gamma\tilde{f}\gamma^{-1}).$$

This follows from the fact that $\eta\tilde{x} = \eta\gamma\tilde{x}$ and

$$\tilde{x} = \tilde{f}(\tilde{x}) \Leftrightarrow \gamma\tilde{f}\gamma^{-1}(\gamma\tilde{x}) = \gamma\tilde{x}.$$

In fact,

$$\eta Fix\tilde{f} = \eta Fix\tilde{f}' \Rightarrow \tilde{f}' = \gamma\tilde{f}\gamma^{-1}$$

for some $\gamma \in Cov\eta$.

Let $[\tilde{f}]$ denote the equivalence class containing $\tilde{f}$ by the relation

$$[\tilde{f}_1] = [\tilde{f}_2] \Leftrightarrow \tilde{f}_2 = \gamma\tilde{f}_1\gamma^{-1}$$

for some $\gamma \in Cov\eta$. Thus different *lifting classes* yield different fixed points of f. In fact, we have

$$Fixf = \bigsqcup_{[\tilde{f}]} \eta Fix\tilde{f},$$

i.e., $Fixf$ is a disjoint union of projections of fixed points of lifts from distinct lifting classes.

11. Reidemeister Classes

By fixing a lift $\tilde{f}$, we obtain a homomorphism $\varphi : Cov\eta \to Cov\eta$ such that

$$\tilde{f}\gamma = \varphi(\gamma)\tilde{f}, \qquad \text{for all } \gamma \in Cov\eta.$$

Since every lift of f is of the form $\alpha\tilde{f}$, we ask when two lifts belong to the same lifting class. Let $\alpha, \beta \in Cov\eta$. Then

$$\begin{aligned}[\alpha\tilde{f}] = [\beta\tilde{f}] &\Leftrightarrow \beta\tilde{f} = \gamma(\alpha\tilde{f})\gamma^{-1} \qquad \text{for some } \gamma \in Cov\eta \\ &\Leftrightarrow \beta\tilde{f} = \gamma\alpha(\tilde{f}\gamma^{-1}) = \gamma\alpha\varphi(\gamma)^{-1}\tilde{f}.\end{aligned}$$

By the uniqueness of lifts, we have

$$[\alpha\tilde{f}] = [\beta\tilde{f}] \Leftrightarrow \beta = \gamma\alpha\varphi(\gamma)^{-1} \tag{11.1}$$

for some $\gamma \in Cov\eta$.

Furthermore, by choosing appropriate basepoint, φ can be identified with $f_\#$ on π_1 (here we also identify $Cov\eta$ with π_1). From now on, we write π for both $Cov\eta$ and $\pi_1(X)$.

Using (11.1), we define the *Reidemeister action* (of π on π) to be

$$\gamma \bullet \alpha \mapsto \gamma\alpha\varphi(\gamma)^{-1} \tag{11.2}$$

The orbits of the action (11.2) are called the *Reidemeister classes*. It follows from (11.1) that there is a one-to-one correspondence between the lifting classes and the Reidemeister classes.

Remark 11.1. If we choose a different lift $\tilde{f}'$ and thus a different homomorphism φ', we get a bijection between the φ-Reidemeister classes and the φ'-Reidemeister classes so that the cardinality of such sets is constant. Let $R(\varphi)$ be the cardinality of the set of φ-Reidemeister classes. It is called the *Reidemeister number* of φ. The Reidemeister number of the map f, denoted by $R(f)$ is simply $R(\varphi)$ or the cardinality of the sset of lifting classes.

11.1. Relationship with $N(f)$. Let $\tilde{f}$ be a lift of f to the universal cover. Suppose $\tilde{x}_1, \tilde{x}_2 \in Fix\tilde{f}$ project to different fixed points x_1, x_2 respectively. Choose a path $\tilde{C} : [0,1] \to \tilde{X}$ such that $\tilde{C}(0) = \tilde{x}_1, \tilde{C}(1) = \tilde{x}_2$. Then $C := \eta \circ \tilde{C} : [0,1] \to X$ is a path from x_1 to x_2 and the path $\tilde{f} \circ \tilde{C}$ projects to the path $f \circ C$ since $\eta(\tilde{f} \circ \tilde{C}) = f \circ \eta \circ \tilde{C} = f \circ C$. In fact, the loop $\tilde{C}(\tilde{f} \circ \tilde{C})^{-1}$ projects to the loop $C(f \circ C)^{-1}$. Since $\tilde{X}$ is simply-connected, $\tilde{C}(\tilde{f} \circ \tilde{C})^{-1}$ is trivial in π_1 and thus $C(f \circ C)^{-1}$ is homotopic to the trivial loop, i.e., $f \circ C \sim C$. In other words, x_1 and x_2 are Nielsen equivalent.

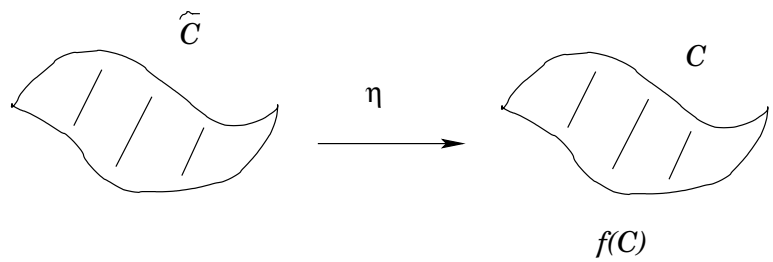

FIGURE 11

Conversely, suppose $x_1, x_2 \in Fixf$ are Nielsen equivalent, i.e., there exists a path C from x_1 to x_2 such that $f \circ C \sim C$. Let $\tilde{f}$ be a lift of f and $\tilde{x}_1 \in Fix\tilde{f}$ such that $\eta(\tilde{x}_1) = x_1$. Lift C to a path $\tilde{C}$ starting at $\tilde{x}_1$ and ending at $\tilde{x}_2$. Then $\tilde{f} \circ \tilde{C}$ projects onto $f \circ C$ which is homotopic to C. Thus, $\tilde{f} \circ \tilde{C}$ also ends at $\tilde{x}_2$ and hence $\tilde{f}(\tilde{x}_2) = \tilde{x}_2$.

If we let $\mathcal{N}(f)$ and $\mathcal{R}(f)$ denote the set of Nielsen classes and the set of Reidemeister classes respectively, then our discussion above shows that there is an injection

$$\mathcal{N}(f) \rightarrowtail \mathcal{R}(f)$$

which implies that $N(f) \leq R(f)$. Now, it is appropriate to distinguish the Nielsen (non-empty) classes from the Reidemeister classes. We call $\eta Fix\tilde{f}$ a *fixed point class* of f. It might be empty. This class corresponds to the lifting class $[\tilde{f}]$ and the Reidemeister class $[1]$ since we fix the lift $\tilde{f}$ and thereby considering the φ-Reidemeister classes.

We should point out that $R(f)$ need not be finite while $N(f) < \infty$. For example, if $f = 1_X$ then any two points (fixed by 1_X) are Nielsen equivalent so $N(f) \leq 1$. On the other hand, the Reidemeister classes are simply the conjugacy classes. In particular, if $\pi_1(X)$ is abelian then $R(1_X) = |\pi_1(X)|$.

12. Computation of the Nielsen Number

We begin with the question: *How do we compute the Nielsen number?*

There are some easy cases: if $f = 1_X$ or if X is simply-connected then there is only one Nielsen class so $N(f) \leq 1$. In these situations, $L(f) = 0 \Rightarrow N(f) = 0$ or $L(f) \neq 0 \Rightarrow N(f) = 1$. Of course, if $f = 1_X$ then $L(f) = \chi(X)$. Thus, the Nielsen number does not give more information than the Lefschetz number.

In 1942, W. Franz showed that if X is the classical lens space then for any selfmap $f : X \to X$, the Nielsen classes have the same fixed point index. In fact, we have

(1) $L(f) = 0 \Rightarrow N(f) = 0$ in which case $f \sim f'$ with $Fix f' = \emptyset$;

(2) $L(f) \neq 0 \Rightarrow N(f) = R(f) = \#Coker(1 - f_{*1})$.

Let's first recall the definition of the classical lens spaces. Consider the 3-sphere

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 | |z_1|^2 + |z_2|^2 = 1\}.$$

Let $\mathbb{Z}_p$ be the cyclic group of order p, p odd prime. Then $\mathbb{Z}_p$ acts freely on S^3 via

$$\zeta \bullet (z_1, z_2) \mapsto (\zeta z_1, \zeta z_2)$$

where $\zeta = e^{2\pi i/p}$. The orbit space $S^3/\mathbb{Z}_p =: L_p$ is the lens space with $\pi_1(L_p) \cong \mathbb{Z}_p$. The canonical map $\eta : S^3 \to L_p$ is a regular p-fold cover.

Consider a lift $\tilde{f}$ so that

$$\begin{array}{ccc} S^3 & \xrightarrow{\tilde{f}} & S^3 \\ {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle\eta} \\ L_p & \xrightarrow[f]{} & L_p \end{array}$$

is commutative. For any $\alpha \in Cov\eta$, $\eta \circ \alpha\tilde{f} = f \circ \eta$. Since $\deg \eta = p \neq 0$, it follows that $\deg(\alpha\tilde{f}) = \deg f$. Thus, $\deg(\alpha\tilde{f})$ is independent of α. By a classical theorem of H. Hopf, we conclude that $\alpha\tilde{f} \sim \tilde{f}$ for any α. This homotopy then implies that $I(f, \eta Fix\alpha\tilde{f}) = I(f, \eta Fix\tilde{f})$ so that if one class is essential all other classes are essential and $L(f) \neq 0 \Rightarrow N(f) = R(f)$.

If one class is inessential then all classes are inessential, thus $L(f) = 0 \Rightarrow N(f) = 0$.

(See figures below.)

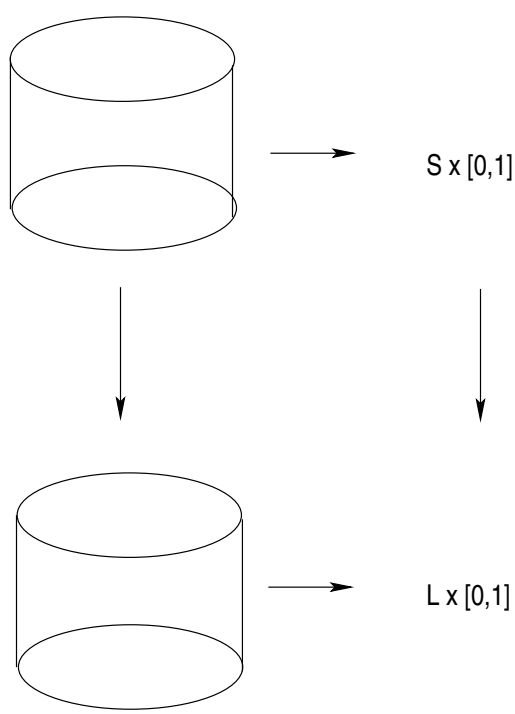

FIGURE 12

The fact that $R(f) = \#Coker(1 - f_{*1})$ follows from the fact that π_1 is abelian and thus isomorphic to H_1. More precisely, since $\pi_1(X) = H_1(X)$, $\gamma\alpha\varphi(\gamma)^{-1} = \bar{\gamma}+\bar{\alpha}-f_{*1}(\bar{\gamma}) = \bar{\alpha}+(1-f_{*1})(\bar{\gamma})$. The Reidemeister action now becomes

$$\bar{\alpha} \mapsto \bar{\alpha} + (1 - f_{*1})(\bar{\gamma})$$

or

$$H_1(X) \to H_1(X)/\mathrm{Im}(1 - f_{*1}) = Coker(1 - f_{*1}).$$

This phenomenon lead B. Jiang (1963) to giving the so-called Jiang conditions under which all fixed point classes have the same fixed point index.

Define

$$J(X) := \{\sigma \in \pi | \sigma\alpha = \alpha\sigma \text{ for all } \alpha \in \pi \text{ and } \sigma \sim 1_{\tilde{X}}\}.$$

By definition, $J(X)$ is a central subgroup of $\pi \equiv Cov\eta \equiv \pi_1(X)$. A space X is called a *Jiang space* if $J(X) = \pi$. Let $\tilde{f}$ be a lift of f. For any $\alpha \in \pi$, if $J(X) = \pi$ then $\alpha \sim 1_{\tilde{X}}$ and so $\alpha\tilde{f} \sim \tilde{f}$. Hence, the fixed point classes $\eta Fix\alpha\tilde{f}$ and $\eta Fix\tilde{f}$ have the same fixed point index. Therefore, we have the following result.

THEOREM 12.1. *Suppose a compact connected polyhedron X is a Jiang space. Then for any selfmap $f : X \to X$,*

$$\begin{aligned} &(1) \quad L(f) = 0 \Rightarrow N(f) = 0; \\ &(2) \quad L(f) \neq 0 \Rightarrow N(f) = R(f) = \#Coker(1 - f_{*1}). \end{aligned} \tag{12.1}$$

Jiang spaces include (1) simply-connected spaces; (2) lens spaces (or generalized lens spaces); (3) Lie groups and H-spaces; (4) coset spaces G/G_0 where G is a compact connected Lie group and G_0 is a connected subgroup.

While the class of Jiang spaces contain many well-known spaces, any X with non-abelian fundamental group cannot be a Jiang space.

Since our ultimate goal is to compute $N(f)$, the effect of the Jiang condition can be achieved if we can assure that the fixed point classes

are either all essential (or the same sign) or all inessential. With this in mind, B. Jiang considered the following slight generalization.

THEOREM 12.2. *Let X be a compact connected polyhedron with finite fundamental group $\pi \equiv Cov\eta$. If for each $\alpha \in \pi$, the induced homomorphism $\alpha_* : H_*(X;\mathbb{Q}) \to H_*(X;\mathbb{Q})$ is the identity then for any $f : X \to X$, (1) $L(f) = 0 \Rightarrow N(f) = 0$; (2) $L(f) \neq 0 \Rightarrow N(f) = R(f)$.*

PROOF. Let $\tilde{f}$ be a lift of f. Under the hypothesis, $(\alpha\tilde{f})_* = \tilde{f}_*$. It follows that $L(\alpha\tilde{f}) = L(\tilde{f})$. Note that the universal cover $\tilde{X}$ is also a compact connected polyhedron. The fixed point set $Fix\tilde{f}$ projects to a fixed point class $\mathbb{F}$ of f. Furthermore, $L(\tilde{f})$ is an integral multiple of $I(f,\mathbb{F})$ such that $L(\tilde{f})$ and $I(f,\mathbb{F})$ have the same sign. We conclude that all fixed point classes have fixed point indices of the same sign. $\square$

When π is infinite and non-abelian, the general computational problem of the Nielsen number remains challenging. Recent progress has been made for coset spaces of compact connected Lie groups as well as certain $\mathcal{C}$-nilpotent spaces where $\mathcal{C}$ is the class of finite groups.

We call X a *Jiang-type* space if the conclusion of Theorem 12.2 holds for all selfmaps of X.

In 1984, D. Anosov showed that for all selfmaps $f : N \to N$ of a compact nilmanifold N, $N(f) = |L(f)|$. In fact, one can show that nilmanifolds are Jiang-type spaces. By a compact nilmanifold, we mean a coset space G/Γ where G is a connected, simply-connected nilpotent Lie group and Γ is a discrete subgroup so that G/Γ is compact. The simplest such example is that of the 3-dimensional Heisenberg manifold.

EXAMPLE 12.1. Let

$$G = \{\left(\begin{smallmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{smallmatrix}\right) | a, b, c \in \mathbb{R}\}$$

and

$$\Gamma = \{\left(\begin{smallmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{smallmatrix}\right) | a, b, c \in \mathbb{Z}\}.$$

The group operation in G is the usual matrix multiplication. As a group, G is nilpotent and non-abelian. As a topological space, G is

homeomorphic to $\mathbb{R}^3$. The coset space G/Γ is a 3-dimensional compact nilmanifold. The same construction in dimension 2 yields the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. Here, $\pi_1(G/\Gamma) \cong \Gamma$ is finitely generated torsion-free nilpotent.

The Anosov theorem has since prompted many new generalizations and further investigations which play an active role in current research in topological fixed point theory.

Remark 12.1. The group $J(X)$ is also known as the (first) Gottlieb subgroup.

So far, we have seen that the fundamental group plays an important role in Nielsen fixed point theory. Thus the computation must necessarily involve π_1 and thus the (universal) covering space. On the other hand, the Lefschetz number is homological in nature and is the sum of all fixed point indices of fixed point classes. Therefore, it is desirable to find some invariant that would incorporate both the homological (Lefschetz) and the geometric (Nielsen) data.

Lecture IV

13. The Reidemeister Trace

In 1936, K. Reidemeister studied Nielsen fixed point theory from the algebraic viewpoint using the universal cover. We shall define a trace-like quantity which captures both $N(f)$ and $L(f)$.

Let X be a compact connected polyhedron and $f : X \to X$ a selfmap. Suppose $\tilde{f} : \tilde{X} \to \tilde{X}$ is a lift of f. First, $\tilde{X}$ inherits a simplicial structure from that of X. The simplicial chain complex $\{C_p(\tilde{X}), \partial\}$ is defined and $C_p(\tilde{X})$ is generated by the oriented p-simplices on $\tilde{X}$.

Again, we identify $\pi = \pi_1 X \equiv Cov\eta$. If $\alpha \in \pi$ and $b \in X$ then $\alpha : \eta^{-1}(b) \to \eta^{-1}(b)$. In fact, for any $\tilde{b}_1, \tilde{b}_2 \in \eta^{-1}(b)$, there exists a unique $\alpha \in \pi$ such that $\tilde{b}_2 = \alpha(\tilde{b}_1)$. We can think of $C_p(\tilde{X})$ as a collection of π-equivariant p-simplices generated by the p-simplices of X. Since X is compact, $C_p(\tilde{X})$ is a finitely generated free $\mathbb{Z}\pi$-module. Here $\mathbb{Z}\pi$ is the integral group ring of π, i.e., a typical element is of the form $\sum n_\alpha \alpha$, $n_\alpha \in \mathbb{Z}, \alpha \in \pi$ and all but a finite number of the n_α's are zero. A $\mathbb{Z}\pi$ basis for $C_p(\tilde{X})$ is a collection of p-simplices $\tilde{\sigma}_1, ..., \tilde{\sigma}_k$ each of which coresponds to a distinct p-simplex σ_i in X. By choosing a $\mathbb{Z}\pi$ basis, the chain map

$$\tilde{f}_{\#p} : C_p(\tilde{X}) \to C_p(\tilde{X})$$

has a matrix $\mathcal{M}_p$ and the trace $\mathrm{tr}\mathcal{M}_p$ is an element of the group ring $\mathbb{Z}\pi$. By fixing the lift $\tilde{f}$, we have a homomorphism $\varphi : \pi \to \pi$. We write $\mathcal{R}_\varphi[\pi]$ as the set of orbits of the Reidemeister action

$$\sigma \bullet \alpha \mapsto \sigma\alpha\varphi(\sigma)^{-1}.$$

Let $\rho : \pi \to \mathcal{R}_\varphi[\pi]$ be the orbit map which extends linearly to $\rho : \mathbb{Z}\pi \to \mathbb{Z}\mathcal{R}_\varphi[\pi]$.

The *Reidemeister trace* of f (also known as the generalized Lefschetz number) with respect to $\tilde{f}$ is given by

$$\mathfrak{L}_\pi(f, \tilde{f}) := \sum_{q=0} (-1)^q \rho \circ \mathrm{tr}\mathcal{M}_q \qquad \in \mathbb{Z}\mathcal{R}_\varphi[\pi]. \tag{13.1}$$

This "trace" is taken at the universal cover level and thus detects the fixed points of lifts of f. Suppose $\tilde{x}$ is a fixed point of $\tilde{f}$ covering the fixed point x of f. Without loss of generality, we may assume that f has finite number of fixed points each of which lies in the interior of a maximal simplex.

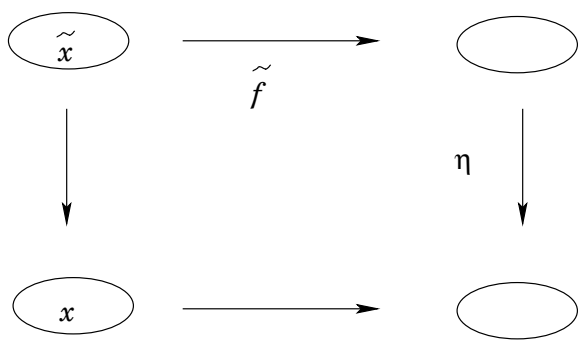

FIGURE 13

Since η is a local homeomorphism, we conclude that

$$I(\tilde{f}, \tilde{x}) = I(f, x).$$

Remember that lifts of the form $\gamma \tilde{f} \gamma^{-1}$ also has fixed points which project to x. Thus, by passing to $\mathcal{R}_\varphi[\pi]$, we would count the contribution to the trace exactly once.

Similar to the Lefschetz-Hopf Theorem, we have the following representation of the Reidemeister trace, due to Wecken.

THEOREM 13.1.

$$\mathfrak{L}_\pi(f, \tilde{f}) := \sum_{\rho \in \mathcal{R}_\varphi[\pi]} i_\rho \rho \qquad i_\rho \in \mathbb{Z}$$

where ρ is the Reidemeister class whose corresponding Nielsen class has fixed point index i_ρ.

Note that if ρ corresponds to an empty Nielsen class then $i_\rho = 0$. The Wecken representation of Theorem 13.1 implies that $N(f) = \#\{\rho | i_\rho \neq 0\}$ and $L(f) = \sum i_\rho$. Therefore, $\mathfrak{L}_\pi(f, \tilde{f})$ captures information on $N(f)$ and on $L(f)$. Furthermore, the Reidemeister trace has the same type of properties as the ordinary Lefschetz number.

14. An Example

Let's compute the Reidemeister trace for a map on the 2-torus.

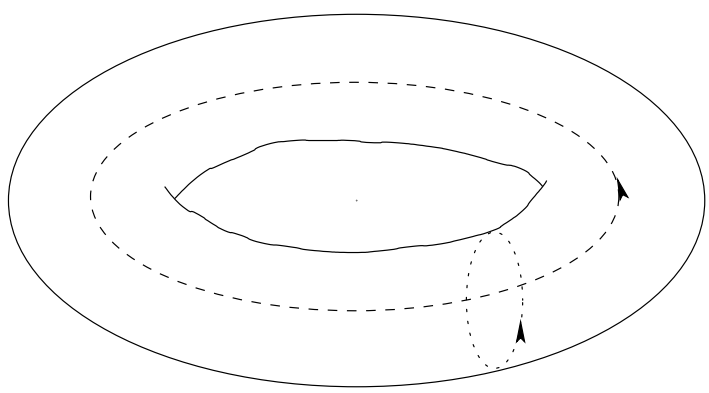

FIGURE 14

Let $T^2 = S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2$. The fundamental group

$$\pi_1(T^2) \cong \mathbb{Z} \oplus \mathbb{Z} = \langle \alpha \rangle \oplus \langle \beta \rangle.$$

Let $f : T^2 \to T^2$ be a map whose induced homomorphism on π_1 is given by φ which sends $\alpha \mapsto \beta^2\alpha^{-1}$ and $\beta \mapsto \beta\alpha^{-1}$. Consider the lift $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$ given by the linear map $\left(\begin{smallmatrix} -1 & -1 \\ 2 & 1 \end{smallmatrix}\right)$. Consider the cells:

$$\tilde{e}_0 = (0,0) \qquad \tilde{e}_1^1 = (s,0), \tilde{e}_1^2 = (0,t)$$

and the square $\tilde{e}_2 = (s,t), 0 \leq s \leq 1, 0 \leq t \leq 1$.

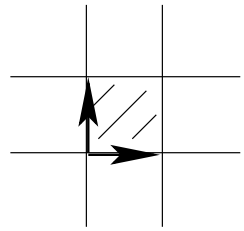

FIGURE 15

Then,

$$\begin{aligned}
\tilde{f}_{\#0}(\tilde{e}_0) &= \tilde{e}_0;\\
\tilde{f}_{\#0}(\tilde{e}_1^1) &= -(\beta^2\alpha^{-1})\tilde{e}_1^1 + (1+\beta)\tilde{e}_1^2;\\
\tilde{f}_{\#0}(\tilde{e}_1^2) &= -\alpha^{-1}\tilde{e}_1^1 + \alpha^{-1}\tilde{e}_1^2;\\
\tilde{f}_{\#0}(\tilde{e}_2) &= (\alpha^{-1} + \alpha^{-1}\beta - \alpha^{-2}\beta^2)\tilde{e}_2.
\end{aligned}$$

To see this, consider the 1-cells:

Since $\begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$,

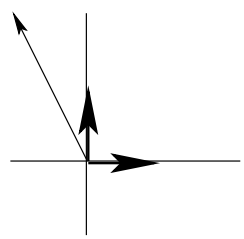

FIGURE 16

we have

$$\tilde{f}_{\#1}(\tilde{e}_1^1) = -(\beta^2\alpha^{-1})\tilde{e}_1^1 + (1+\beta)\tilde{e}_1^2.$$

Similarly, $\begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$,

it follows that

$$\tilde{f}_{\#0}(\tilde{e}_1^2) = -\alpha^{-1}\tilde{e}_1^1 + \alpha^{-1}\tilde{e}_1^2.$$

We determine the images of the 0-cell and the 2-cell in a similar fashion.

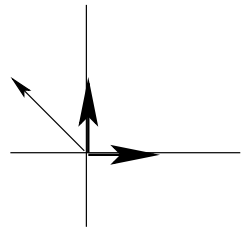

FIGURE 17

Therefore,

$$\mathfrak{L}_\pi(f,\tilde{f}) = [1] - (-[\beta^2\alpha^{-1}] + [\alpha^{-1}]) + ([\alpha^{-1}] + [\beta\alpha^{-1}] - [\beta^2\alpha^{-2}]). \tag{14.1}$$

Now the question is: *Is* (14.1) *the Wecken representation as in Theorem 13.1?*

Next, we need to calculate the Reidemeister classes in $\mathcal{R}_\varphi[\pi]$. First, note that

$$\varphi(\alpha) = \beta^2\alpha^{-1} \quad \text{and} \quad \varphi(\beta) = \beta\alpha^{-1}.$$

Since π is abelian, if $\sigma \in \pi$ then $\sigma = \alpha^m\beta^n$ for some $m, n \in \mathbb{Z}$. Thus,

$$\begin{aligned}\sigma\alpha\varphi(\sigma)^{-1} &= \alpha^m\beta^n\alpha\varphi(\alpha^{-m}\beta^{-n}) \\ &= \alpha^m\beta^n\alpha\beta^{-2m}\alpha^m\beta^{-n}\alpha^n \\ &= \alpha^{n+1}\beta^{-2m}\end{aligned}$$

or $\sigma\varphi(\sigma)^{-1} = \alpha^n\beta^{-2m}$.

This means that the unity 1 and any element of the form $\alpha^n\beta^{\text{even}}$ are in the same Reidemeister class. Hence,

$$\mathcal{R}_\varphi[\pi] = \{[1], [\beta]\} \cong \mathbb{Z}_2.$$

Moreover,

$$[\beta^2\alpha^{-1}] = [1] = [\alpha^{-1}] = [\alpha] = [\beta^2\alpha^{-2}]$$

and

$$[\beta\alpha^{-1}] = [\beta].$$

We conclude that

$$\mathfrak{L}_\pi(f,\tilde{f}) = [1] + [\beta]$$

which implies that $L(f) = 2$ and $N(f) = 2$.

In the case of an orientable surface of genus $g > 1$, $X = \sum^g$, $\pi_1(X)$ is generated by $2g$ elements satisfying one relation. Using the Fox derivative on integral group rings, E. Fadell and S. Husseini compute the Reidemeister trace and showed that

$$\mathfrak{L}_\pi(f, \tilde{f}) = \rho(1 - \sum_i \frac{\partial f_\#(a_i)}{\partial a_i} + A)$$

where $(\frac{\partial f_\#(a_i)}{\partial a_j}) = J(\varphi)$ is the Jacobian matrix and $A \in \mathbb{Z}\pi$ such that $A(\nabla R) = \varphi(\nabla R)J(\varphi)$, R being the relation.

The most difficult part of the computation is the ability to express the Reidemeister trace in the Wecken representation, i.e., the ability to distinguish Reidemeister classes.

15. Application to Dynamics

In discrete dynamical systems, given a homeomorphism $f : X \to X$, one studies the periodic points of f, i.e., points x such that $f^n(x) = x$ for some positive integer n. In general, dynamicists are interested in the long term behavior of a dynamical system and hence the periodic behavior asymptotically. Instead of studying $Fixf^n$ for a fixed n, we'd like to study *all* of the periodic points all at once. To do so, we must be able to *view* all the periodic orbits in one place.

Let X be a topological space and $f : X \to X$ a map. The *mapping torus* of f is the quotient space

$$T_f := X \times [0, \infty)/(x, s + 1) \sim (f(x), s)$$

for $x \in X, s \in [0, \infty)$.

There is a natural *semi-flow* on T_f given by the map $\varphi : T_f \times [0, \infty) \to T_f, \varphi_t(x, s) = (x, s + t), \forall t \geq 0$.

If $x \in Fixf^n$ then we obtain a closed orbit in T_f which crosses the section $X \times \{0\}$ n times (not distinct unless n is minimal). Now, let $\Gamma := \pi_1(T_f)$. Then Γ is an extension of $\pi = \pi_1 X$ by $\mathbb{Z}$. In fact, if f is

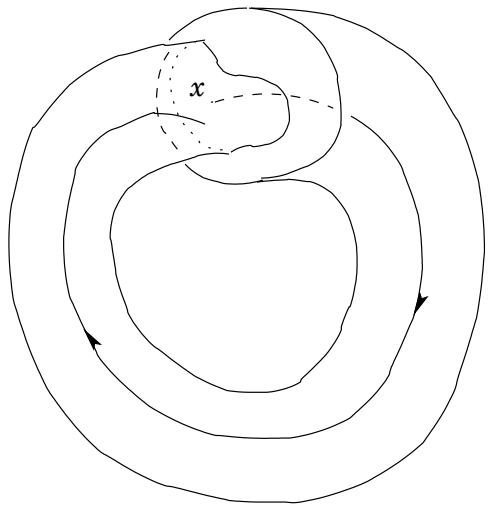

FIGURE 18

a homeomorphism, then Γ has the following presentaion:

$$\Gamma = \langle \pi, z | xz = zf_{\#}(x), x \in \pi \rangle.$$

For every dimension q, the cellular chain map $\tilde{f}_{\#q}$ gives rise to a $\mathbb{Z}\pi$-map $\tilde{F}_q$ and hence a matrix with respect to a given $\mathbb{Z}\pi$ basis. Now,

$$x\alpha f_{\#}(x)^{-1} = \beta \quad \Leftrightarrow \quad x(\alpha z^{-1})x^{-1} = \beta z^{-1}$$

i.e., αz^{-1} and βz^{-1} are conjugate. This means that the twisted conjugation (via $f_{\#}$) in π becomes ordinary conjugation in Γ. Equivalently, one can consider conjugacy classes in Γ instead of Reidemeister classes in π. Let Γ_c be the set of conjugacy classes in Γ. We can then define

$$\mathfrak{L}_\Gamma(f, \tilde{f}) := \sum_{q=0} (-1)^q \rho_\Gamma \circ \mathrm{tr}(z\tilde{F}_q) \qquad \in \mathbb{Z}\Gamma_c \tag{15.1}$$

where $\rho_\Gamma : \Gamma \to \Gamma_c$ is the canonical projection.

Similarly, for the iterates f^n, we have

$$\mathfrak{L}_\Gamma(f^n, \tilde{f}^n) := \sum_{q=0} (-1)^q \rho_\Gamma \circ \mathrm{tr}(z\tilde{F}_q)^n \qquad \in \mathbb{Z}\Gamma_c. \tag{15.2}$$

Note that *all* the Reidemeister traces for various n belong to the *same* abelian group $\mathbb{Z}\Gamma_c$. In particular for homeomorphism on surfaces, Fox derivative can be used to compute $\mathfrak{L}_\Gamma(f^n, \tilde{f}^n)$.

For example, if X is the punctured disk then X has the same homotopy type of a wedge of circles. In this case,

$$\mathfrak{L}_\Gamma(f^n, \tilde{f}^n) = [1] - [\mathrm{tr}(zD)^n]$$

where

$$D = (\frac{\partial f_\#(a_i)}{\partial a_j}).$$

Here, there are no 2-dimensional cells and the circles in the wedge yield the generators $\{a_j\}$.

Remark 15.1. Nielsen fixed point theory and Reidemeister trace have been used to study multiple solutions of differential equations and to the study of flows in dynamics.

16. Conclusion

There are two central issues in Nielsen fixed point theory:

1. (realization) Does there exist $f' \sim f$ such that $N(f') = MF[f]$?

2. (computation) How do we compute $N(f)$?

Here,

$$MF[f] := \min\{\#Fixg | g \sim f\}.$$

For compact manifolds X we have two cases for issue **1.**: (i) $\dim X \neq 2$: Wecken's theorem 9.1 gives a positive answer. (ii) When $\dim X = 2$ the answer is NO if $\chi(X) < 0$. In fact, for any positive integer m, one can find a map $f_m : X \to X$ such that

$$MF[f_m] - N(f_m) \geq m.$$

So the real question in dimension two is: How do we compute $MF[f]$?

For issue **2.**, there is the Jiang-condition but it poses severe restriction on the fundamental group and hence on the space.

Sometimes, a space X is made up by gluing together simpler pieces along ("parametrized by") a "simple" space or it is covered by a nice space. For example, if $\hat{X}$ is a regular cover of a space X and $f : X \to X$ is a selfmap. One can sometimes compute or estimate $N(f)$ in terms of $N(\hat{f})$ where $\hat{f}$ varies over the set of lifts of f. Another example is fiber-preserving maps of a fibration $p : X \to B$ (e.g., $X = S^3$, $B = S^2$ and $p : S^3 \to S^3/S^1$ is the Hopf fibration). Suppose $f : X \to X$ is a fiber map such that

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow[\bar{f}]{} & B \end{array}$$

is commutative. If $b \in Fix\bar{f}$ then f induces a selfmap $f_b : F \to F$ where $F = p^{-1}(b)$. Under certain conditions, $N(f)$ can be computed in terms of $N(\bar{f})$ and $N(f_b)$.

Jiang-type spaces also provide examples for which the Nielsen number can be computed using the Reidemeister number.

We end these notes by discussing one particular variation or generalization of fixed point theory, namely root theory, which has received much attention in recent research. For other related topics, we refer the reader to the conference proceedings [**4**], [**3**], [**9**], [**11**], [**7**], and [**6**].

Shortly after S. Lefschetz announced his fixed point theorem, he generalized his result to the coincidence setting. Let $f, g : X \to Y$ be maps between closed connected orientable triangulated n-manifolds. Using Poincaré duality, a coincidence Lefschetz number $L(f, g)$ was defined. Then he showed that $L(f, g) \neq 0$ implies that for any $f' \sim f, g' \sim g$, $Coin(f', g' := \{x \in X | f'(x) = g'(x)\}$ must be non-empty. In the 1940s, W. Franz generalized Nielsen's notion of fixed point classes to coincidences. Two points $x, y \in Coin(f, g)$ are *Nielsen equivalent* (as coincidences of f, g) if there exists a path C from x to y such that $f \circ C \sim g \circ C$ relative to the endpoints. By defining the coincidence index, the Nielsen number, which is simply the number of coincidence

classes with non-zero coincidence index, can be defined. Franz's student H. Schirmer, proved in her 1955 thesis a coincidence analog of Wecken's theorem 9.1, showing that for $n \geq 3$, the Nielsen number is a sharp lower bound for the minimal number of coincidence points in the homotopy class of f and of g. In one special case, when g is a constant map, the coincidence theory becomes *root* theory, i.e., the study of the solutions of the equation $f(x) = a$. In fact, a Nielsen type theory was already developed by H. Hopf in 1931 in his work on the so-called Hopf degree theory. One special feature about roots is the following theorem due to Hopf (also rediscovered independently and in a more general setting by R. Brooks in 1973).

THEOREM 16.1. *Let $f : X \to Y$ be a map between two closed connected orientable manifolds and $a \in Y$. If $\deg f = 0$ then $N(f; a) = 0$. If $\deg f \neq 0$ then $N(f; a) = [\pi_1 Y : f_\#(\pi_1 X)]$ where $N(f; a)$ denotes the Nielsen root number of f with respect to a.*

Hopf's theorem is a Jiang-type result in the sense that all root classes have the same root index. When $\deg f \neq 0$, $N(f)$ coincides with the Reidemeister number $R(f; a)$ which is equal to the index of the subgroup $f_\#(\pi_1 X)$ in $\pi_1 Y$. In other words, root theory is more computable and in many situations it can be used to obtain information about fixed points or coincidences.

To illustrate this point, let's consider a selfmap $f : T^2 \to T^2$ on the 2-torus. We associate to f a map $\varphi_f : T^2 \to T^2$ given by $\varphi_f(x) = x^{-1} \cdot f(x)$. Here, we regard the torus T^2 as a topological group so that x^{-1} denotes the inverse of x in T^2 and $x^{-1} \cdot f(x)$ is the product (group operation) of the elements x^{-1} and $f(x)$ in T^2. Note that $f(x) = x$ iff $\varphi_f(x) = e$, the group unity in T^2. In other words, the fixed points of f become the roots of φ_f. As it turns out, the Nielsen theory for fixed points of f can be translated into the Nielsen theory (after Hopf) for roots of φ_f. Moreover, one can show that $L(f) = \deg \varphi_f$. Thus, it is easy to see that the result of Theorem 16.1 can be re-written as that in Theorem 12.1.

Appendix

17. Review

In this appendix, we recall a few basic facts from elementary algebraic topology.

Given two topological spaces X and Y, a homotopy equivalence from X to Y is a map $f : X \to Y$ together with a map $g : Y \to X$ such that $g \circ f \sim 1_X$ and $f \circ g \sim 1_Y$. In this case, we say that X and Y have the same homotopy type and we write $X \sim Y$.

In $\mathbb{R}^{n+1}$, we let $e_i = (0, ..., 0, 1, 0, ..., 0)$ where the 1 occurs in the i-th coordinate. The points $e_1, ..., e_{n+1}$ span the *standard n-simplex* Δ_n. In other words,

$$\Delta_n := \{(x_1, ..., x_{n+1}) | x_i \geq 0 \text{ and } \sum x_i \leq 1\}.$$

For instances, Δ_0 is just a point, Δ_1 is the line segment between $(0, 1)$ and $(1, 0)$,

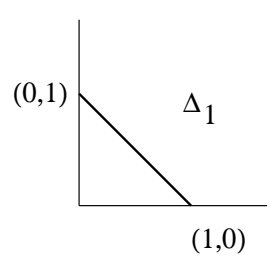

FIGURE 19

and Δ_2 is the equilateral triangle whose vertices are $(0, 0, 1), (0, 1, 0), (1, 0, 0)$.

A *(singular) p-chain* on a topological space X is simply a continuous map $\sigma_p : \Delta_p \to X$. Let $S_p(X) = S_p(X; \mathbb{Z})$ be the free abelian group

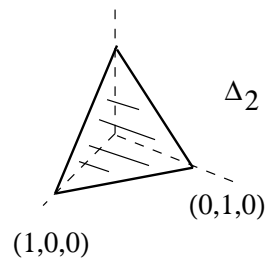

FIGURE 20

generated by the set of p-chains. Define the *boundary operator* $\partial_p : S_p \to S_{p-1}$ by

$$\partial_p(\sigma_p)(\Delta_{p-1}) = \sum_{i=1}^{p}(-1)^i \sigma_p(x_1, ..., \widehat{x_i}, ..., x_p)$$

where $\widehat{x_i}$ means that we omit the i-th coordinate. It turns out that $\partial_p\partial_{p+1} = 0$ so that $\{S_p(X), \partial_p\}$ is a *chain complex*.

Now, we let $Z_p := \mathrm{Ker}\partial_p$ and $B_p := \mathrm{Im}\partial_{p+1}$. Note that $B_p \subseteq Z_p$. The *singular p-th homology group of X* is given by

$$H_p(X) = H_p(X; \mathbb{Z}) := Z_p/B_p.$$

If (X, A) is a pair, we define $S_p(X, A) = S_p(X)/S_p(A)$. The boundary operators on $S_p(X)$ and on $S_p(A)$ induce an operator ∂ on $S_p(X, A)$. It follows that $\{S_p(X, A), \partial\}$ is a chain complex and its homology, denoted by $H_*(X, A)$, is called the singular relative homology of (X, A).

For simplicial complexes or cellular complexes, we define the corresponding chain complexes and hence the corresponding homology groups. For a compact polyhedron X, we have

$$H_*^{\text{singular}}(X) \cong H_*^{\text{simplicial}}(X) \cong H_*^{\text{cellular}}(X).$$

If $f : X \to Y$ and $g : Y \to Z$ are maps, then f induces a homomorphism $f_{*q} : H_q(X) \to H_q(Y)$. Moreover, we have $(g \circ f)_* = g_* \circ f_*$.

Bibliography

[1] Robert F. Brown, "The Lefschetz Fixed Point Theorem," Scott-Foresman, Glenview, Illinois, 1971.

[2] Robert F. Brown, Fixed point theory. *History of topology*, 271–299, North-Holland, Amsterdam, 1999.

[3] Fixed point theory and its applications. Proceedings of a conference held at the International Congress of Mathematicians, University of California, Berkeley, California, August 4–6, 1986. Edited by R. F. Brown. Contemporary Mathematics, 72. American Mathematical Society, Providence, RI, 1988.

[4] Fixed point theory. Proceedings of a Conference held at the Université de Sherbrooke, Sherbrooke, Quebec, June 2–21, 1980. Edited by Edward Fadell and Gilles Fournier. Lecture Notes in Mathematics, 886. Springer, Berlin, 1981

[5] D.L. Gonçalves and J.C. de Souza Kiihl, "Teoria do Índice," 14° Colóquio Brasileiro de Matemática, IMPA, Rio De Janeiro, 1985.

[6] Theory of fixed points and its applications. Proceedings of the International Conference held at the University of São Paulo, São Paulo, July 26–30, 1999. Edited by Daciberg L. Gonçalves and Peter N. Wong. Topology Appl. 116 (2001), no. 1. Elsevier Science B.V., Amsterdam, 2001.

[7] Nielsen theory and Reidemeister torsion. Papers from the Workshop on Dynamical Zeta Functions, Nielsen Theory and Reidemeister Torsion held in Warsaw, June 24–July 5, 1996. Edited by Jerzy Jezierski. Banach Center Publications, 49. Polish Academy of Sciences, Institute of Mathematics, Warsaw, 1999.

[8] B. Jiang, "Lectures on Nielsen Fixed Point Theory," Contemp. Math. **14**, Amer. Math. Soc., Providence, 1983.

[9] Topological fixed point theory and applications. Proceedings of the conference held in Tianjin, April 5–8, 1988. Edited by Bo Ju Jiang. Lecture Notes in Mathematics, 1411. Springer-Verlag, Berlin, 1989.

[10] T. Kiang, "The Theory of Fixed Point Classes," Springer, New York, 1987.

[11] Nielsen theory and dynamical systems. Proceedings of the AMS-IMS-SIAM Summer Research Conference held at Mount Holyoke College, South Hadley,

Massachusetts, June 20–26, 1992. Edited by Christopher K. McCord. Contemporary Mathematics, 152. American Mathematical Society, Providence, RI, 1993

[12] J. Vick, "Homology Theory," Academic Press, New York 1973.