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ON WHITEHEAD PRODUCTS, FOX GROUPS, GOTTLIEB GROUPS, AND RHODES GROUPS

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ABSTRACT. In this lecture, we re-introduce the so-called torus homotopy groups of R. Fox. We generalize Fox's construction and obtain generalized Fox groups in which the generalized Whitehead product of M. Arkowitz is again a commutator. The generalized Gottlieb group is shown to be central in this generalized Fox group. Finally, we extend this construction to transformation groups, generalizing a result of F. Rhodes. Details will appear elsewhere.

1. INTRODUCTION

This paper is a write-up of the lecture given by the third author at the *International Conference on Homotopy Theory and Related Topics* held at the Korea University in Seoul, South Korea during Febraury 01 - 04, 2005.

In classical Nielsen fixed point theory (see e.g., [10]), the fundamental group plays a crucial role. One of the first computational tools in computing the Nielsen number is the concept of the so-called Jiang subgroup of the fundamental group. Furthermore, D. Gottlieb used Nielsen fixed point theory to establish his theorem on the triviality of the center of the fundamental group of an aspherical complex of non-zero Euler characteristic. In fact, the Jiang subgroup is the same as the first Gottlieb group. In [8], D. Gottlieb introduced the evaluation subgroups of higher homotopy groups and these subgroups are now known as the Gottlieb groups.

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In studying Nielsen fixed point theory for equivariant mappings, the notion of an equivariant Jiang subgroup was introduced in [16]. This has since been generalized to higher equivariant Gottlieb groups in [5]. Since deck transformations are homeomorphisms of the universal cover covering the identity map, one considers naturally the following group

$$\hat{G} := \{\hat{\gamma} \in \mathrm{Homeo}(\tilde{X}) | \eta\tilde{\gamma} = \gamma\eta \text{ for some } \gamma \in G\},$$

where $\eta : \tilde{X} \to X$ is the universal covering of a G-complex X for a finite group G. Identifying $\pi_1(X)$ with the group of deck transformations, it is easy to see that

$$1 \to \pi_1(X) \to \hat{G} \to G \to 1$$

is a short exact sequence.

It turns out that $\hat{G}$ is isomorphic to $\sigma_1(X, G)$, the first Rhodes group of the transformation group (X, G) first defined in [12]. In fact, higher Rhodes groups $\sigma_n(X, G)$ had been defined in [13]. Unlike $\sigma_1(X, G)$ which is an extension of $\pi_1(X)$, the higher Rhodes groups $\sigma_n(X, G)$ are not extensions of $\pi_n(X)$ by G. Instead, the following sequence

$$1 \to \tau_n(X) \to \sigma_n(X, G) \to G \to 1$$

is exact. Here, $\tau_n(X)$ denotes the n-th torus homotopy group first introduced by R. Fox in [3] and further studied in [4]. The connection between Nielsen fixed point theory and the first Rhodes group has led us to our further investigation of the Fox groups and of the Rhodes groups. This results in our work in [6] and [7].

This lecture reports some of the results obtained in [6] and in [7]. For simplicity, all spaces in this paper are assumed to be well-pointed and compactly generated.

2. Fox torus homotopy groups and their generalizations

The classical Whitehead product [15] of $\alpha \in \pi_m(X)$ and $\beta \in \pi_n(X)$ is an element $[\alpha, \beta]$ in $\pi_{m+n-1}(X)$. In the special case when $m = n = 1$, $[\alpha, \beta]$ is simply the commutator. In [3] and in [4], R. Fox gave a geometric interpretation of the Whitehead product. In doing so, he introduced a bigger group in which the Whitehead product is again a commutator.

Definition 2.1. Let X be a space and $x_0 \in X$. Denote by $T^r = S^1 \times \cdots \times S^1$ (r-copies) the r-dimensional torus. For any positive integer n, the n-th Fox (torus homotopy) group of X is defined to be

$$\tau_n(X, x_0) = \pi_1(X^{T^{n-1}}, \overline{x_0}),$$

where $\overline{x_0}$ is the constant map at x_0. Here $X^{T^{n-1}}$ is the space of continuous maps from T^{n-1} to X with the usual compact open topology.

We should point out that $X^{T^{n-1}}$ is *not* the space of basepoint preserving maps from T^{n-1} to X, which is not even of the same homotopy type. In particular when $n = 2$, X^{S^1} denotes the free loop space ΛX and not the loop space ΩX. Moreoever, $\tau_1(X, x_0) = \pi_1(X, x_0)$.

R. Fox showed in [4] that $\tau_n(X, x_0)$ is completely determined by the homotopy groups $\pi_i(X, x_0)$ for $i \leq n$ and their Whitehead products. More precisely, he proved the following result.

Theorem 2.2. *Let X be a path connected space and $x_0 \in X$. For any positive integer $n \geq 2$, the sequence*

$$0 \to \prod_{i=2}^{n} \pi_i(X)^{\alpha_i} \to \tau_n(X) \overset{\leftarrow}{\to} \tau_{n-1}(X) \to 1$$

is split exact, where α_i is the binomial coefficient $\binom{n-2}{i-2}$.

Since τ_1 coincides with π_1, it follows from this result that $\tau_n(X, x_0)$ is not abelian in general. In [4], R. Fox showed that the Whitehead product in π_{m+n-1} of two elements $\alpha \in \pi_m, \beta \in \pi_n$ is a commutator when π_{m+n-1} is embedded in τ_k for $k \geq m + n - 1$.

Fox's construction shows that, for example in the case $n = 2$, $\tau_2(X, x_0)$ can be interpreted as follows. Let F_2 be the pinched two dimensional torus, i.e., F_2 is the quotient of $S^1 \times S^1$ by $S^1 \times \{s_0\}$ for some basepoint $s_0 \in S^1$. Then $\tau_2(X, x_0) \cong [F_2, X]$, the set of homotopy classes of basepoint preserving maps from F_2 to X. In fact,

$$\tau_2(X, x_0) = [\Sigma(S^1 \sqcup *), X] \quad \text{or more generally}$$

$$\tau_n(X, x_0) = [\Sigma(T^{n-1} \sqcup *), X].$$

Let $F_n = T^n/T^{n-1}$ be the (n-dimensional) pinched torus. It follows that $F_n \approx \Sigma(T^{n-1} \sqcup *)$. One can show that

$$\Sigma F_{n-1} \approx \bigvee_{i=2}^{n} (S^i)^{\binom{n-2}{i-2}}.$$

With this re-interpretation, we re-establish Fox's result as follows.

Theorem 2.3. *Let X be a path connected space and $x_0 \in X$. For any positive integer $n \geq 2$, the sequence*

$$0 \to \tau_{n-1}(\Omega X) \to \tau_n(X) \overset{\leftarrow}{\to} \tau_{n-1}(X) \to 1$$

is split exact. Furthermore,

$$\tau_{n-1}(\Omega X) \cong \prod_{i=2}^{n} \pi_i(X)^{\alpha_i}.$$

With respect to a fibration $F \hookrightarrow E \to B$, we have the following

Theorem 2.4. *For any positive integer k, there is a long exact sequence*

$$\cdots \to \tau_k(\Omega^n F) \to \tau_k(\Omega^n E) \to \tau_k(\Omega^n B) \xrightarrow{d_*} \tau_k(\Omega^{n-1} F) \to \cdots.$$

Now, it is straightforward to generalize τ_n as follows.

Definition 2.5. Let X be a space and $x_0 \in X$. For any space W, the W-Fox group of X is defined to be

$$\tau_W(X, x_0) = [\Sigma(W \sqcup *), X].$$

It is clear that τ_W reduces to τ_n when $W = T^{n-1}$. Moreover, if we let $W = S^{n-1}$ then $\tau_W(X, x_0) = \kappa_n(X, x_0)$, the n-th Abe group of X studied in [1].

Our main result in [6] is the following generalization of Fox's result (Theorem 2.3).

Theorem 2.6. *For any path connected X, V and W, the following sequence is split exact.*

$$1 \to [(V \times W)/V, \Omega X] \to \tau_{V \times W}(X) \overset{\leftarrow}{\to} \tau_V(X) \to 1$$

If $W = \Sigma A$ is a suspension, then $[(V \times W)/V, \Omega X]$ is abelian and is isomorphic to $[V \wedge W, \Omega X] \times [W, \Omega X]$.

3. Generalized Whitehead products and generalized Gottlieb groups

In [8], D. Gottlieb introduced the evaluation subgroups of the higher homotopy groups, now known as the Gottlieb groups. More precisely, the n-th Gottlieb group $G_n(X)$ of a space X is given by

$$G_n(X) := \text{Im } (ev_* : \pi_n(X^X, 1_X) \to \pi_n(X, x_0)),$$

where ev is the evaluation map. It is well known that G_1 is central in π_1 and for any $\alpha \in G_n$ and for any $\beta \in \pi_m$, the Whitehead product $[\alpha, \beta] = 0$. By replacing π_n with τ_n, it is natural to ask whether the evaluation subgroup is abelian and how it is related with the classical Gottlieb groups.

With respect to τ_n, we define the n-th Gottlieb-Fox group to be

$$G\tau_n(X) := \text{Im } (ev_* : \tau_n(X^X, 1_X) \to \tau_n(X, x_0)).$$

In [6], we obtained the following result.

Theorem 3.1. *The Gottlieb-Fox group is a direct product of ordinary Gottlieb groups. More precisely, for any positive integer n,*

$$G\tau_n(X) = \prod_{i=1}^{n} G_i(X)^{\gamma_i},$$

where γ_i is the binomial coefficient $\binom{n-1}{i-1}$ and $G_i(X)$ is the i-th Gottlieb group of X. Furthermore, $G\tau_n(X)$ is central in $\tau_n(X)$.

The Gottlieb groups have been generalized to very general settings. K. Varadarajan [14] defined the generalized Gottlieb group $\mathcal{G}(\Sigma A, X)$ as a subgroup of the group $[\Sigma A, X]$ to be

$$\mathcal{G}(\Sigma A, X) := \text{Im}(ev_* : [\Sigma A, (X^X, 1_X)] \to [\Sigma A, (X, x_0)]).$$

In [11], K. Lim showed that $\mathcal{G}(\Sigma A, X)$ is central in $[\Sigma A, X]$.

Similar to the classical Gottlieb groups, elements of the generalized Gottlieb group $\mathcal{G}(\Sigma A, X)$ have trivial generalized Whitehead products with elements in any $[\Sigma B, X]$.

Recall that for any $\alpha \in [\Sigma A, X]$ and any $\beta \in [\Sigma B, X]$, M. Arkowitz [2] introduced the generalized Whitehead product of α and β to be the element

$[\alpha, \beta] = [K']$, where $K' : \Sigma(A \wedge B) \to X$ is induced by the map

$$K := f'^{-1} \cdot g'^{-1} \cdot f' \cdot g' : \Sigma(A \times B) \to X$$

which, when restricted to $\Sigma A \vee \Sigma B$, is homotopic to a constant. Here, $f : \Sigma A \to X$, $g : \Sigma B \to X$ are two maps representing α and β respectively. The composites $f' = f \circ \Sigma p_A, g' = g \circ \Sigma p_B$, where $\Sigma p_A : \Sigma(A \times B) \to \Sigma A$ and $\Sigma p_B : \Sigma(A \times B) \to \Sigma B$, are maps from $\Sigma(A \times B)$ to X. Using the co-multiplication of $\Sigma(A \times B)$, K is a well-defined map.

In the spirit of [4], the following from [6] can be regarded as a reinterpretation of Arkowitz's generalized Whitehead product.

Theorem 3.2. *Given $\alpha \in [\Sigma A, X]$ and $\beta \in [\Sigma B, X]$, the image of $[\alpha, \beta]$ in $\tau_{A\times B}(X)$ is the commutator of the image of α^{-1} and the image of β^{-1} in $\tau_{A\times B}(X)$.*

Using the projection $A \times B \to A$, we can regard $[\Sigma A, X]$ as a subgroup of $[\Sigma(A\times B), X]$. Under this identification, we have the following generalization of a result of C. Hoo who showed in [9] that $\mathcal{G}(\Sigma A, X)$ is central in $[\Sigma A, X]$.

Theorem 3.3. *The generalized Gottlieb group $\mathcal{G}(\Sigma A, X)$, regarded as a subgroup of $\tau_{A\times B}(X)$, is central in $\tau_{A\times B}(X)$ for any B. In particular, it is central in $[\Sigma A, X]$.*

4. Rhodes groups and their generalizations

Let G be a finite group and X be a G-complex. Choose a basepoint $x_0 \in X$. In [12], F. Rhodes extended the definition of the fundamental group of a space to that of a transformation group (X, G). He then defined higher homotopy groups of (X, G) in [13]. For any positive integer $n \geq 1$, denote by $C_n = \{(t_1, \hat{t}_2, \ldots, \hat{t}_n)\}$ denote the n-dimensional cylinder, where $0 \leq t_i \leq 1$ and $\hat{t}_i = t_i \pmod 1$. Recall that a map $f : C_n \to X$ is of order $g \in G$ provided $f(0, \hat{t}_2, \ldots, \hat{t}_n) = x_0$ and $f(1, \hat{t}_2, \ldots, \hat{t}_n) = g(x_0)$. Denote by $[f; g]$ the homotopy class of the map f of order g. Then the operation via

$$[f_1; g_1] * [f_2; g_2] := [f_1 + g_1 f_2; g_1 g_2].$$

gives the set $\sigma_n(X, x_0, G)$ of homotopy classes of maps of order $g \in G$ a group structure. The following was proved by F. Rhodes in [13].

Theorem 4.1. *For any positive integer n, the sequence*

$$1 \to \tau_n(X, x_0) \hookrightarrow \sigma_n(X, x_0, G) \to G \to 1$$

is exact.

This result in the case $n = 1$ was obtained earlier in [12].

In [18], M. Woo and Y. Yoon considered the evaluation subgroup

$$\mathcal{G}_n := \mathcal{G}_n(X, x_0, G) := \mathrm{Im}(ev_* : \sigma_n(X^X, 1_X, G) \to \sigma_n(X, x_0, G)).$$

They asked whether $\mathcal{G}_n$ is necessarily abelian. In [7], we showed the following

Theorem 4.2. *Let $\mathcal{W}_n$ denotes the n-th Gottlieb-Fox subgroup, i.e., $\mathcal{W}_n = G\tau_n(X)$. Then the sequence*

$$1 \to \mathcal{W}_n \to \mathcal{G}_n \to G_0 \to 1$$

is exact, where G_0 is the subgroup of G consisting of elements g each of which lies in the same component as 1_X.

This result led us to construct an example of a non-abelian $\mathcal{G}_n$ (in fact, for the case $n = 1$).

In section 2, we generalized the Fox torus homotopy groups τ_n. Now, we generalize the Rhodes homotopy groups σ_n in a similar way.

Definition 4.3. For any space W with a basepoint w_0 and any G-space X with basepoint x_0, we let

$$\sigma_W(X, x_0, G) = \{[f; g] | f : \widehat{\Sigma} W, w_0 \to X, x_0\},$$

where $[f; g]$ denotes the homotopy class of the map f of order $g \in G$ and $\widehat{\Sigma} W$ denotes the un-reduced suspension, i.e., $\widehat{W} = W \times [0, 1]/\sim$, where $(w_1, 0) \sim (w_2, 0)$ and $(w_1, 1) \sim (w_2, 1)$. Under the operation $[f_1; g_1] * [f_2; g_2] := [f_1 + g_1 f_2; g_1 g_2]$, $\sigma_W(X, x_0, G)$ is a group.

Generalizing Rhodes result (Theorem 4.1), we have the following

Theorem 4.4. *For any space W, the sequence*

$$1 \to \tau_W(X, x_0) \to \sigma_W(X, x_0, G) \to G \to 1$$

is exact.

For more connections with the equivariant Gottlieb groups of [5], with equivariant Nielsen fixed point theory, and properties of σ_n and of σ_W, we refer the reader to [7].

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