Two examples of circular motion for introductory courses in relativity

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The circular twin paradox and Thomas precession are presented in a way that makes them accessible to students in introductory relativity courses. Both are discussed by examining what happens during travel around a polygon and then in the limit as the polygon becomes a circle. Because relativistic predictions based on these examples are verified in experiments with macroscopic objects (such as atomic clocks flown in airplanes and the gyroscopes on Gravity Probe B), they are especially convincing to introductory students.
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I. INTRODUCTION

Experimental confirmations of relativistic effects are especially important for students in introductory relativity courses. Each provides a particular situation that makes the abstract more concrete, and gives students a physical framework in which to understand what is (and is not) happening.

One of the most frequently discussed relativistic effects is the twin paradox. In its most common form Twin A stays on Earth[1] while Twin B travels at a constant velocity **v** to a neighboring star. Twin B then moves into a new inertial frame and travels with a velocity **v** back to Earth. The paradox is formulated by making two predictions that cannot both be true. On the one hand, because Twin B is moving in the frame of Twin A, Twin A should measure a clock held by Twin B to run slow. On the other hand, because Twin A is moving in the frame of Twin B, Twin B should measure a clock held by Twin A to run slow. The paradox is that when the two twins meet, their clocks are face to face and each cannot have run slow relative to the other.

The generally acknowledged resolution to the paradox is that since Twin B had to accelerate in order to return to Earth, Twin B had to change inertial frames while Twin A didn't. Consequently, an accelerometer can distinguish between the two twins, and there is no conflict with the postulate of special relativity if the clock carried by Twin B runs slowly relative to the clock carried by Twin A and not vice versa.

Following their presentation of the twin paradox many textbooks discuss its experimental confirmation in cyclotrons (see, for example, Refs. 2 and 3) or with atomic clocks flown in airplanes (see, for example, Refs. 4 and 5). However, neither experimental arrangement involves clocks moving along linear trajectories. Rather, they more closely resemble what is called the "circular twin paradox," which is formulated as follows: Suppose Twin A and Twin B are each on different rings and that the rings are rotating in opposite directions about a common axis through their centers. One ring can either be just above the other or right next to it. Assume each twin is holding a clock and that both clocks read zero when the twins first meet. The paradox is the same as in the linear case: Because Twin A is moving relative to Twin B, and Twin B is moving relative to Twin A, each should measure the other's clock as running slow compared to their own. What distinguishes the circular twin paradox from the standard twin paradox is that now both twins accelerate in the same manner. It is true that one accelerates in the clockwise and the other in the counterclockwise direction, but because time dilation depends only on the magnitude of the velocity the different directions can't matter. Consequently, unlike what happens in the standard twin paradox, the magnitude of acceleration measured on an accelerometer can't distinguish between the two travelers. Just as in the standard twin paradox, when the clocks meet and are face to face, each cannot have run slow relative to the other. There can be only one answer; what will that answer be?

Lightman, *et al.*[6] gave a short, formal solution to the circular twin paradox and Cranor, Heider, and Price[7] considered the paradox in more detail. One purpose of Ref. 7 was to "present an analysis that should be both mathematically and physically intelligible to beginning relativity students."[8] Their approach involved a detailed analysis of four different ways in which clocks on a rotating ring could be synchronized and a discussion of the consequences of each synchronization method. The analysis became somewhat complex, but the authors said the complexity was necessary because, in their opinion, "Special relativistic time dilation must be considered in the context of clock synchronization…," and "Time dilation will only be observed from a reference frame in which the clocks are appropriately synchronized…[8]."

In this paper we show that the circular twin paradox can be resolved more simply by using the results of the linear twin paradox and that, contrary to what is claimed in Ref. 7, it is not necessary to examine clock synchronization in the reference frame of either twin. Our approach is to first consider what happens when the twins travel on concentric polygons and then to take the limit as the polygons tend to a circle.

The polygon method also can be used to discuss Thomas precession,[9] which, as far as we know, is not included in any introductory treatment of special relativity. By using elementary arguments involving rotations and boosts, we show that relative to an inertial observer, the coordinate axes of a reference frame traveling around a polygon are rotated when they return to their starting point by an angle of $2\pi(\gamma-1)$ relative to their initial orientation (where, as usual, $\gamma = 1/\sqrt{1-v^2/c^2}$ and c is the speed of light). Surprisingly, the direction of the rotation depends on whether the polygon is traversed in the clockwise or counterclockwise direction. In fact, Thomas precession is one of the few effects in special relativity that depends on the direction of the velocity rather than only on its magnitude. (Other examples occur in the addition of noncollinear velocities.[10,11])

The first experimental confirmation of Thomas precession occurred just after the concept of spin was proposed.[12] In fact, Thomas precession played an important role in validating both spin and quantum mechanics since, without Thomas precession, nonrelativistic quantum theory is unable to correctly predict the fine structure of the spectrum of hydrogen, hydrogen-like atoms, alkali atoms, etc.

Perhaps less well known is that another experiment to measure Thomas precession has just been completed. The Stanford-NASA satellite Gravity Probe B (GP-B), launched in 2004, contains four gyroscopes predicted to precess, in part, due to Thomas precession. Consequently, Thomas precession is especially interesting to students because it involves the confirmation of a prediction of special relativity with a macroscopic object, and because it is part of a contemporaneous physics research experiment they can understand. In April, 2007, the GP-B group announced that their preliminary gyroscope precession data agreed with the predicted value of Thomas precession to within an experimental uncertainty of less than 1%.[13]

After discussing the major predictions of relativity, most texts discuss their experimental confirmation with muons, pions, and other elementary particles. Many students in introductory relativity courses are unfamiliar with these elementary particles, and thus come to think of relativity as an interesting theory that only applies to the subatomic world and has little to do with their everyday life. Many are surprised to learn about relativistic effects on macroscopic objects such as atomic clocks and gyroscopes. Studying experimental confirmations made with these objects causes many students to take the concepts of relativity more seriously and to develop a deeper appreciation of their importance.

The outline of this paper is as follows: In Sec. II we review the standard twin paradox and another paradox based on it. In Sec. III we discuss the circular twin paradox, its relation to the standard twin paradox, and its resolution. In Sec. IV we discuss predictions based on the circular twin paradox and their confirmation in experiments with elementary particles in cyclotrons and macroscopic atomic clocks flown in airplanes. In Sec. V we discuss how acceleration can be treated in the various twin paradoxes. In Sec. VI we derive Thomas precession and show how it affects a gyroscope orbiting the Earth. In Sec. VII we discuss the confirmation of Thomas precession in an experiment carried out on the satellite Gravity Probe B. We then use the gyroscope analogy to discuss spin and the experimental confirmation of Thomas precession in the spectrum of hydrogen. In Sec. VIII we summarize our results and discuss where they fit into an introductory relativity course.

The main body of this paper is written so that it can be understood by students in relativity courses for nonmajors, such as those using the texts by Mermin[14] or Baierlein.[15] The more quantitative parts have been removed from each section and placed in one of the three appendices. In this sense the paper has two tracks, one designed for a more conceptually based course and the other for courses in which students have worked with Lorentz transformations and the relativistic form of Newton's second law; that is, for courses such as the modern physics part of a calculus-based introductory course or any of the higher level courses in the physics curriculum.

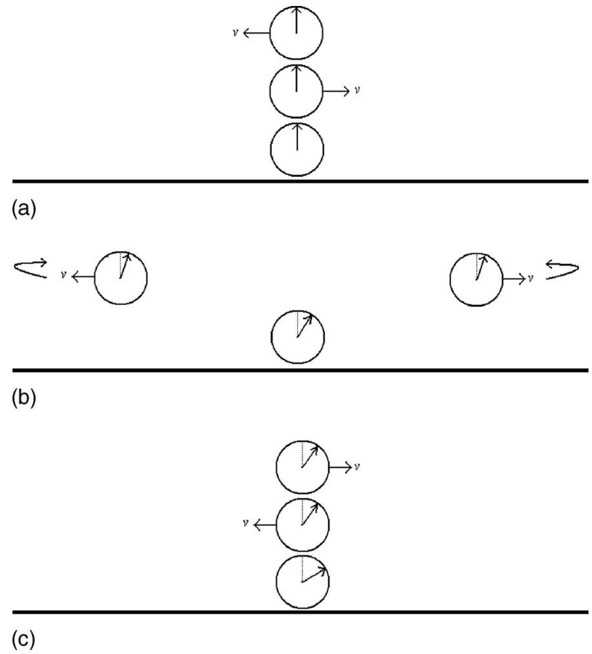

Fig. 1. The triplet paradox, in which one triplet remains on Earth, one travels to the right with a speed v, and the other to the left with the same speed v. Both of the traveling triplets move the same distance away from the Earth, turn around, and then return. The question is: When the traveling triplets meet, how will their clocks compare with each other and with a clock held by the Earth triplet.

II. LINEAR TWIN PARADOXES

To understand the circular twin paradox it is helpful to first consider a slight generalization of the standard twin paradox to what can be called the "triplet" paradox. Imagine triplets, one stationed on the Earth,[1] one traveling with a speed v to the right, and the other traveling with a speed v to the left (see Fig. 1). After each of the two traveling triplets has gone the same distance away from the Earth, each moves into another inertial frame and travels back to the Earth with the same speed v. Each triplet has a clock and the question is: When the two traveling triplets meet, how will their clocks compare. Just as in the standard twin paradox each traveler can argue that the other traveler's clock should run more slowly than their own. However, in the triplet paradox both travelers undergo the same magnitude of acceleration so the reading on an accelerometer can no longer distinguish between them. The direction of the accelerations cannot matter because time dilation depends only on the magnitude of the velocity and not on its direction. Therefore, the argument used to resolve the standard twin paradox won't resolve the triplet paradox.

The triplet paradox can be resolved by decomposing it into two twin paradoxes. Just as in the standard twin paradox, when each of the traveling triplets compares their clock with the clock on Earth, each will agree that their clock is running slower than the Earth clock by the same factor of γ.[16] Consequently each of the travelers will agree that the other traveler's clock is reading exactly the same time as their own and thus the paradox is resolved. It is interesting to note that in this approach the resolution is not obtained by directly comparing the clocks held by the two traveling triplets; rather, it comes from comparing each of the traveling clocks with a third clock, the one that remains on Earth. (Stated more sim-

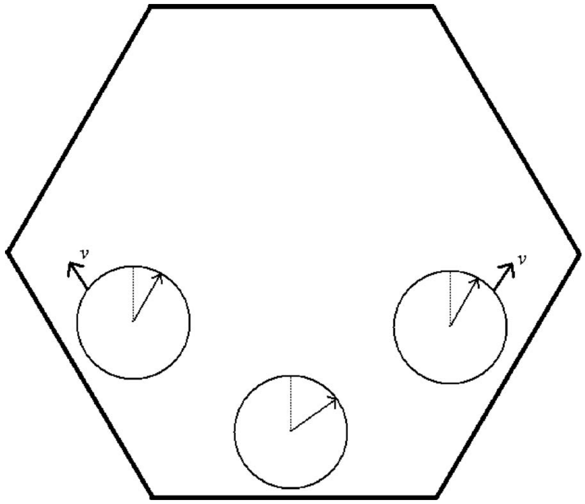

Fig. 2. Two triplets travel on a hexagonal path away from the Earth-based triplet. Again, the question is: When the traveling triplets meet, how will their clocks compare with each other and with a clock held by the Earth observer.

ply, in this approach the triplet paradox is resolved by the transitive relation: If $A=C$ and $B=C$, then $A=B$. In our case A and B are the traveling clocks and C is the clock that remains on Earth.)

III. THE CIRCULAR TWIN PARADOX

The circular twin paradox can be resolved with the same method used in the triplet paradox. Consider a polygon with N sides, as shown in Fig. 2. Again consider the triplets, but now with each of the traveling triplets moving with a speed v along the polygon trajectory, one in the clockwise and the other in the counterclockwise direction. Assume that when the traveling triplets first pass the Earth-based triplet, all three clocks are compared and all read zero. Along the side of the polygon at the bottom of Fig. 2 the clock carried by each of the traveling triplets runs slow relative to the Earth clock by a factor of γ. As the clocks move along each successive side of the polygon they continue to run slow by a factor of γ, so when the travelers are again across from the Earth triplet, each of their clocks will be running slow by the same factor of γ relative to the Earth-based clock. Consequently, when they are again across from the Earth clock, the clocks held by each of the traveling triplets must read exactly the same. All that remains is to take the limit as N goes to infinity. In this limit the polygon becomes a circle and all of the results obtained for the polygon path are true for the circular path. Note that this result is obtained by direct comparison of the traveler's clocks with the clock held by the Earth observer so there is no need to introduce a system of synchronized clocks in any of the reference frames involved.

It is interesting that from the point of view of a traveling triplet, the polygon path they follow is different from the polygon path measured by the Earth triplet. If the Earth triplet measures each side of the polygon to be of length L_0/N, the traveler will measure each side to be of length $L_0/(N\gamma)$. Consequently, the traveling triplet will measure the total distance around the polygon to be less than that measured by the Earth observer. Furthermore, if the Earth observer says that as $N\to\infty$ the polygon becomes a circle of circumference $L_0=2\pi R$, the traveling triplet will say that in this limit the polygon goes to a circle of circumference $L_0/\gamma=2\pi R/\gamma$. Because lengths perpendicular to the motion don't contract, both triplets will agree that the radius of the circle is R. Consequently, because the ratio of the circumference of their circle to its radius is less than 2π, the traveler will conclude that the geometry of their frame is non-Euclidean.[17] Einstein used a similar argument to prove that accelerating frames will have non-Euclidean geometries, and thus, that a theory of relativity generalized to include accelerated frames (general relativity) will have to be constructed using non-Euclidean geometry.[18] Thus the circular twin problem also can be used to show students an important aspect of general relativity and to give them an idea of what is meant by curved (or "warped") space.

IV. EXPERIMENTAL CONFIRMATIONS OF THE CIRCULAR TWIN PARADOX

The first experimental confirmation of the time dilation predicted in the circular twin paradox was reported by Hay *et al.* in 1960.[3] They put a Co^{57} "clock" on the surface of a cylinder whose radius was 0.4 cm, and a Fe^{57} receiver diametrically opposite to it at a radius 6.64 cm. When they rotated the unit at various angular speeds up to 500 rev/s (corresponding to a linear speed of 7×10^{-7} c), they found the predicted and measured time dilations agreed to within an experimental uncertainty of 2%.[19]

In 1979, Bailey *et al.*[2] performed a more accurate measurement using positive and negative muons traveling in circular orbits at the Muon Storage Ring at CERN. The speed of these muons was approximately $0.9994c$ ($\gamma\approx29.3$). They measured a time dilation of the mean half lives that agreed with what was predicted by special relativity to within an experimental uncertainty of 0.1%. Note that the experiments reported in Refs. 2 and 3 involve the same configuration as the circular twin paradox, which makes the paradox especially appropriate for discussing these two confirmations.

One of the advantages of including the circular twin paradox in an introductory relativity course is that predictions based on it have been confirmed with atomic clocks flown in airplanes. Students who are only just learning about elementary particles are more impressed and convinced by experiments with macroscopic objects. In 1971, Hafele and Keating[4] synchronized four cesium atomic clocks with a reference clock at the United States Naval Observatory in Washington, DC and then flew them around the Earth in commercial jets. The clocks were flown first from east to west and then from west to east. (Four clocks were used on each flight in order to measure the average time elapsed and the experimental uncertainty of the average.) When the clocks returned to the Naval Observatory, they were compared with a reference clock. The time difference predicted for the westward flight was $275\pm8\%$, and the observed time difference (averaged over the four clocks) was 273 ns$\pm3\%$. The overall experimental uncertainty is calculated by considering the standard deviation in each of the two results, and is usually expressed by saying that the observed and measured values agree to within ten percent.[20]

Although the experiment of Hafele and Keating sounds simple enough when presented in this way, there are several complications which are sometimes worth discussing in class. First, the predicted and experimentally measured discrepancies between the Earth-based and flying clocks actually arise from two effects: time dilation in special relativity, and a time dilation from general relativity which predicts that clocks at different heights in a gravitational field will run at different rates. The fact that it is the total time dilation that is confirmed experimentally doesn't alter the experimental confirmation of the time dilation predicted by special relativity. Rather, the agreement between the predicted and observed time dilations confirms the predictions of *both* special and general relativity.

Second, because commercial airplanes were used, the ground speed, latitude, longitude and altitude were not constant during each flight. The westward flight was divided into 108 intervals and the values of these four variables were recorded at the appropriate times. The time dilation predicted by special relativity was then obtained from these 108 data sets by numerical integration, which is one of the reasons there is an uncertainty associated with the predicted result.

There is one additional detail that is sometimes worth discussing in class or assigning as a student project. Because the Earth is rotating while the traveling clocks are in the air, its rotational speed also must be taken into account. Hafele and Keating present a clear and easy-to-follow discussion of this point.[21] The result is that the relative velocity of the reference clock and the clock flown in the direction of the Earth's rotation will differ from that of the reference clock and the clock flown in the direction opposite to the Earth's rotation. The difference predicted for the eastward flight was -40 ± 23 ns, and the observed difference was -59 ± 10 ns. Although these numbers don't agree quite as well as the ones for the westward flight, they still confirm time dilation to within the associated uncertainties.

It is interesting to note that the time difference between the Earth clock and the clocks flown eastward is negative, while the difference between the Earth clock and the clocks flown westward is positive. This difference in sign means the clocks on the westward flight ran faster than the Earth clock. At first, the fact that a clock moving relative to the surface of the Earth runs faster than a clock fixed on the Earth is somewhat surprising. However, consider what is happening from the reference frame of an inertial observer fixed in space on the axis of the Earth above the North Pole. This observer will observe the reference clock on Earth moving in a circular orbit eastward with a speed equal to the linear speed of a point on the surface of the Earth (at the location of the Naval Observatory). This observer also will see the clock in the airplane moving in a circular orbit, but westward with a speed that is the resultant of the speed of the flying clock with respect to the Earth and the Earth's speed with respect to the inertial frame. If the plane were traveling with a velocity relative to the inertial frame exactly equal but opposite to the Earth's linear velocity of rotation, then the inertial observer would see exactly the configuration of the circular twin paradox and conclude that both clocks should read the same time whenever they meet. In this way it is easy to understand what Hafele and Keating[4] show explicitly: If the clock flown in the westward direction is flying faster than the Earth's linear speed of rotation, it will run slower than the clock fixed on Earth (the difference will be negative), whereas if the clock flown westward is flying slower than the linear speed of rotation of the Earth, then the Earth clock will run slower than the clock in the airplane (the difference will be positive). Because of the actual speeds of the jets used in the experiment, the clocks flown westward ran faster than the reference clock on Earth.

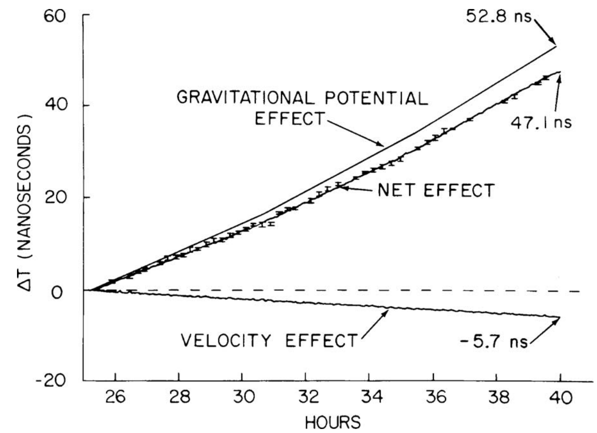

Fig. 3. Data from Ref. 5 showing their experimental confirmation of time dilation (Ref. 23). The vertical axis shows the difference between atomic clocks on Earth and atomic clocks flown in a "racetrack" path over Chesapeake Bay. The line labeled "velocity effect" is the time dilation predicted by special relativity, the line labeled "potential effect" is the time dilation predicted by general relativity, and the line labeled "net effect" is the sum of these two time dilations. The dots show the experimental data which, as can be seen, are in excellent agreement with the theoretical prediction.

In contrast, the clock flown eastward, in the direction of the Earth's rotation, will always be flying faster than the linear speed of rotation of the Earth, and so it will always run slower than the reference clock on Earth.

A more accurate experiment with atomic clocks was performed by Alley *et al.* in 1975.[5] This group put six atomic clocks, three cesium beam clocks and three rubidium gas cell clocks, in a U.S. Navy P3C anti-submarine patrol plane, which made five 15-h flights in an elliptical ("racetrack") path over Chesapeake Bay. Just as in the experiment of Hafele and Keating[4] the ground speed, altitude, etc., of the plane were recorded, but this time continuously with both *X*-band and *C*-band radar so the integral of the time dilation could be calculated more accurately. After landing, the plane was parked alongside a group of six identical reference clocks so a direct comparison could be made. The clocks flown in the plane ran slower than the clocks that remained on the Earth. The magnitude of the predicted difference was 47.1 ns$\pm$0.5% and the magnitude of the measured difference was 46.5 ns$\pm$1.6%. A plot of their results in which the prediction of special relativity (the "velocity effect"), a prediction of general relativity (the "gravitational potential effect"), and the net prediction is shown in Fig. 3.[23] The agreement between the observed and predicted time dilations is remarkable,[24] and confirms the predictions of both special and general relativity.

V. ACCELERATION IN THE TWIN PARADOXES

There still is the question of whether we are justified in ignoring acceleration in the twin, triplet, and circular twin paradoxes. In other words, does the rate of an ideal clock depend only on its speed relative to an inertial frame, and is it independent of the clock's acceleration? By "ideal clock" we mean a clock based on nuclear or particle decay rates, on the wavelength (or frequency) of an atomic transition, etc.

The experiments in Refs. 2 and 3 provide an excellent context in which to discuss this question. In both cases ideal clocks were held in circular orbits by an applied magnetic field so their speed was constant. That is, even though the clocks were accelerating, γ remained constant. Therefore, if the time dilation predicted by special relativity agrees with the time dilation found experimentally, then the rate of an ideal clock has been shown experimentally to depend only on its speed and to be independent of its acceleration.

As mentioned, Hay *et al.*[3] found that when a Co^{57} clock was put on a rotating wheel, the predicted and measured time dilations agreed to within an experimental uncertainty of 2%. They also reported that their clocks experienced a constant acceleration of magnitude 10^4 times the acceleration of gravity. Consequently, the experiment of Hay *et al.* confirms that for accelerations of this magnitude, the rate of an ideal clock depends only on its speed and is independent of its acceleration to within an experimental uncertainty of 2%.[27]

When measuring the time dilation of the mean lifetime of muons in the Muon Storage Ring, Bailey *et al.*[2] reported that the muons experienced accelerations of 10^{18} times the acceleration of gravity. (We show in Appendix A that this value can be calculated from the parameters of the experiment.) Consequently, their experiments confirm that for accelerations of this magnitude, the rate of an ideal clock depends only on its speed and is independent of its acceleration to within an experimental uncertainty of 0.1%. Therefore, experiments confirm that we are justified in ignoring the effect of acceleration on time dilation in the circular twin problem.

The question of acceleration in the other paradoxes can be handled in the same way. If we assume that in the linear paradoxes each traveling twin is inside a charged vehicle moving with a constant velocity away from the Earth, then they can pass into an inertial frame returning to the Earth by being accelerated in a semicircle by an externally applied magnetic field. If the travelers accelerate in this way, then the experiments we have discussed confirm that the time dilation of their clocks depends only on the constant value of γ and is independent of their acceleration. Similarly, when the twins are traveling around an N-gon, a uniform magnetic field can accelerate their charged vehicle in a small circular arc from one side of the N-gon to the next. The greater the magnetic field, the smaller the arc. The time over which the acceleration occurs also can be minimized by increasing the strength of the magnetic field. As N increases, the change of direction $2\pi/N$ at each vertex decreases and the total change in direction (2π) is independent of N.

Using these results we can summarize the various paradoxes from the point of view of the Earth observer, that is, the observer who always remains in one inertial frame. In the standard twin paradox one twin moves away from the Earth and during this part of the trip the Earth observer measures the traveler's clock to run slow due to time dilation. The traveling twin then accelerates into another reference frame heading back toward the Earth. Experiments confirm that the acceleration necessary to change frames has no affect on the rate of the traveler's (ideal) clock. On the return trip the Earth observer again measures the traveler's clock to run slow due to time dilation. Consequently, when the traveler returns to Earth, the Earth observer predicts that the traveler will be younger, and that the traveler's clock will have run more slowly by exactly the amount predicted by time dilation. This analysis extends to the triplet paradox because it is constructed from two standard twin paradoxes, and to the circular twin paradox because what happens in this case is the result of what happens as the traveler moves from one straight line segment of the N-gon to the next, which is what

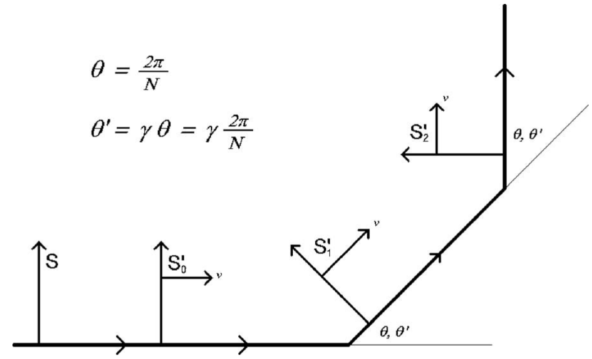

Fig. 4. A reference frame S'_0 moves along the first segment of an N-gon with a speed v. If v is nonrelativistic then the frame must rotate through an angle θ to align its x' axis with the next segment of the N-gon and become frame S'_1. If v is relativistic, then the frame can only align its x' axis with the next segment if it rotates through an angle of $\theta' = \gamma\theta$.

happens in the standard twin paradox. Finally, as we have discussed, the time dilation predicted in the circular twin paradox has been confirmed experimentally to a very high degree of accuracy.

One might wonder how the traveler explains why, when she returns to her starting point, more time has elapsed on the Earth clock then on her own. Because students often ask this question, and because including a description of the trip from the traveler's point of view completes the analysis of the circular twin paradox, we discuss it in Appendix B.

VI. THOMAS PRECESSION

We now show that if a traveling twin carries a gyroscope[29] around an N-gon then, after one complete trip, she will observe the axis about which the gyroscope is spinning to have rotated through an angle of $2\pi(\gamma-1)$ relative to its direction at the beginning of the trip.[30] In the limit that the N-gon becomes a circle, the traveling twin will observe the spin axis of the gyroscope to be rotating with an angular speed $\omega_T = 2\pi(\gamma-1)/P$, where P (the period) is the time it takes the twin to complete one trip around the N-gon. The angle through which the spin axis moves relative to its initial direction is called the Thomas rotation angle and the rate ω_T of Thomas rotation is called Thomas precession. It is important to remember that Thomas rotation and Thomas precession are observed by the traveling twin and not by the Earth twin, and that they are as real to the traveling twin as Coriolis and centrifugal forces are to an observer in a rotating frame of reference.

To derive the Thomas rotation and precession, first suppose the traveling twin moves around the N-gon in the counterclockwise direction with a constant nonrelativistic speed v. Also, suppose the traveler keeps realigning her x' axis so that it points along the side of the N-gon on which she is traveling. In order to do this, at each corner she can first rotate her x' axis through an angle θ given by $\tan\theta = 2\pi/N$ and then boost along the new direction. We assume the boost doesn't change the speed of the traveler, only her direction. After one complete trip the traveler's x' axis will have rotated through 2π and will point in exactly the same direction as when the trip began.

Figure 4 shows the trip in more detail. The traveling twin begins at rest with respect to the Earth in the frame S and

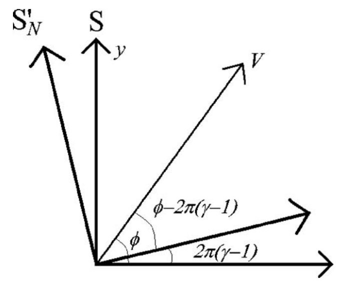

Fig. 5. The spin axis V of a gyroscope makes an angle ϕ with the horizontal axis in the reference frame S. The frame S then moves at a relativistic speed around an N-gon. When S returns to its starting point its axes have rotated in the counterclockwise direction by an angle $2\pi(\gamma-1)$. The new axes are denoted by S_N'. The spin axis V of the gyroscope makes an angle $\phi-2\pi(\gamma-1)$ with the horizontal axis of S_N'.

then boosts into the frame S_0' traveling with a velocity $\vec{v}$ in the horizontal direction. As the traveler approaches the first corner, she rotates her coordinate system and then boosts into a new frame S_1' moving along the second side of the N-gon. As long as she is moving slowly, the angle between the horizontal axes of S_0' and S_1' is θ with $\tan\theta=2\pi/N$. In Fig. 4 we have approximated $\tan\theta$ by θ because our main interest is in what happens as $N\to\infty$ and in this limit the approximation $\tan\theta\approx\theta$ becomes exact.

Now suppose the traveling twin makes the same trip but at a constant relativistic speed v. Because of length contraction, she doesn't agree with her Earth twin that she must rotate her x' axis through $\theta=2\pi/N$ in order to stay on the N-gon. Rather, if the Earth twin measures $\tan\theta=y/x$, then because the traveling twin observes a length contraction $x'=x/\gamma$ in the direction of travel, the angle through which she has to rotate her coordinate system is θ', where $\tan\theta'=y'/x'=y/(x/\gamma)=\gamma(y/x)=\gamma\tan\theta$. This rotation is shown in Fig. 4 in which, as before, we have approximated $\tan\theta'$ by θ'. Therefore, at each corner $\theta'=\gamma\theta=\gamma(2\pi/N)$ as N becomes large.

When the traveling twin makes one (relativistic) round trip and returns to the point at which she began, her x' axis will have rotated by an angle of $2\pi\gamma$. Figure 5 shows the orientation of the final frame S_N' with respect to the initial frame S after one round trip. Consequently the Earth twin says the coordinate system of the traveling twin has rotated in the counterclockwise direction by an angle of $2\pi(\gamma-1)$.

Suppose the traveling twin makes her round trip in a satellite carrying a gyroscope. If the spin axis V of the gyroscope originally made an angle of ϕ with her horizontal axis then, when she returns to her starting point, V will now make an angle $\phi-2\pi(\gamma-1)$ with that axis, as shown in Fig. 5. Because she refers everything to her coordinate system, the traveling twin will observe the spin axis of the gyroscope to have rotated in the clockwise direction by an angle $2\pi(\gamma-1)$ relative to her horizontal axis. This is the Thomas rotation angle.

In the limit $N\to\infty$ the N-gon becomes a circle. Because the Thomas rotation is independent of N, we see that relative to the traveler, the spin axis of the gyroscope she is carrying rotates in the clockwise direction by $2\pi(\gamma-1)$ each time she makes one trip around the circle in the counterclockwise direction. If P is the period of the orbit, then the traveling twin observes the spin axis of the gyroscope to rotate clockwise in the plane of the orbit with an angular speed

$$\omega_T=(2\pi/P)(\gamma-1)=\omega(\gamma-1), \tag{1}$$

where $\omega=(2\pi/P)$ is the angular speed (in radians per second) with which the twin moves around the circle. The angular speed ω_T is called the "Thomas precession."

If students are familiar with the vector description of circular motion (as presented, for example, in Ref. 33 or 34) then, as we show in Appendix C, the angular velocity vector $\vec{\omega}_T$ of Thomas precession can be written in a more general (and standard) form. Nonetheless, as we now show, students need only understand a few basic properties of circular motion in order to derive a simple equation from which the important applications of Thomas precession can be obtained.

Suppose Δs is the arclength of the circle of radius R that is traveled by the satellite when it moves through an angle $\Delta\theta$ in a time Δt. Then $\Delta\theta=\Delta s/R$ and the angular speed of the satellite in its circular orbit is

$$\omega=\frac{\Delta\theta}{\Delta t}=\left(\frac{\Delta s}{R}\right)\frac{1}{\Delta t}=\left(\frac{\Delta s}{\Delta t}\right)\frac{1}{R}=\frac{v}{R}. \tag{2}$$

If we substitute this result into Eq. (1), we have

$$\omega_T=\omega(\gamma-1)=\left(\frac{v^2}{vR}\right)(\gamma-1), \tag{3a}$$

which implies that

$$\omega_T=\left(\frac{va}{v^2}\right)(\gamma-1), \tag{3b}$$

where $a=v^2/R$ is the usual centripetal acceleration.

The reason for writing Eq. (3b) in such a peculiar form is that in the nonrelativistic limit $\gamma\approx 1+v^2/2c^2$ and Eq. (3b) becomes

$$\omega_T=\frac{va}{2c^2}. \tag{4}$$

We now discuss two experimental confirmations of Thomas precession which follow from Eq. (4). First, suppose a satellite is in a circular orbit of radius R about a body of mass M. In this case the acceleration a of the satellite is caused by the gravitational force and $a=GM/R^2$. Because the orbit is circular, $a=v^2/R$, which means that $v=\sqrt{GM/R}$. If the satellite contains a gyroscope, then according to Eq. (4), the twin in the satellite will see the gyroscope precess with angular speed

$$\omega_T=\frac{1}{2c^2}\frac{GM}{R^2}\sqrt{\frac{GM}{R}}=\frac{1}{2c^2R}\left(\frac{GM}{R}\right)^{3/2}. \tag{5}$$

Note that the exact form of the Thomas precession in Eq. (3b) depends on v/c (through γ), as does the nonrelativistic form in Eq. (4). However, for a satellite in a circular orbit about a mass M, Eq. (5) shows that the Thomas precession is independent of the speed and depends only on the radius of the orbit.

As a second example of Thomas precession, consider the electron in a hydrogen atom. In this case the "satellite" is the electron, the "gyroscope" is the electron's spin vector, and the acceleration is caused by the Coulomb force. Because the

Coulomb interaction holds the electron in its orbit, the acceleration in Eq. (4) is $a = ke^2/m_e r^2$, where k is a constant, m_e is the mass of the electron, and e is the charge of the electron. If we replace GM in Eq. (4) with ke^2/m_e, we find that in this case the magnitude of the Thomas precession is

$$\omega_T = \frac{1}{2c^2 r}\left(\frac{ke^2}{m_e r}\right)^{3/2}. \tag{6}$$

VII. EXPERIMENTAL CONFIRMATION OF THOMAS PRECESSION

A. Gyroscopes orbiting the Earth: Gravity Probe B

During the two year period 1959–1960, Pugh[35] and Schiff[36] independently showed that the spin axis of a gyroscope orbiting the Earth in a satellite should precess. They found that three different effects, one from special relativity and two from general relativity, should contribute to the precession: the Thomas effect, the "geodetic effect," which results from a general relativistic correction to Newton's theory of gravity, and the "Lense-Thirring effect," which results from the Earth "dragging" the local inertial frame in the direction of its rotation.[38] If the satellite is in a polar orbit (that is, in an orbit that passes over both the North and South Poles), then the Lense-Thirring precession is in the plane perpendicular to the plane of the orbit and in this way is separated from the Thomas and geodetic precessions, both of which are in the plane of the orbit.[39] Consequently, a measurement of the net precession in the satellite's orbital plane should detect both Thomas and geodetic precession.

Work on designing and implementing such an experiment began in 1962. After the heroic efforts of many people over many years, the satellite Gravity Probe B (GP-B) was launched in 2004.[40] In principle, the experimental setup is simple. Put a satellite containing a gyroscope and an optical telescope in orbit around the Earth. Point both the telescope and the spin axis of the gyroscope toward a guide star. Keep the telescope pointed toward the guide star and measure the direction of the spin axis of the gyroscope over a time span of one year. If the satellite is in a polar orbit, then the movement of the spin axis in the plane of the orbit should be that predicted by the sum of the Thomas and geodetic precessions, and the movement in the plane perpendicular to the plane of the orbit should be that predicted by Lense and Thirring.

The practical obstacles to measuring these precessions become apparent when Eq. (5) is used to calculate their magnitudes. The radius of the GP-B orbit is $r = R_E + r_s = 6.378 \times 10^6$ m $+ 6.493 \times 10^5$ m $= 7.0274 \times 10^6$ m.[41] The mass of the Earth is $M = 5.974 \times 10^{24}$ kg, the speed of light is $c = 2.9979 \times 10^8$ m/s, and the gravitational constant is $G = 6.672 \times 10^{-11}$ N m^2/kg^2. By substituting these values into Eq. (5), we find that $\omega_T = 1.0145 \times 10^{-12}$ rad/s. The geodetic precession is twice this magnitude and in the opposite direction.[39] Consequently, the total precession in the plane of the orbit is predicted to be 3/2 the magnitude of the Thomas precession, or $0.001834°$/yr = 6.6 arcsec/yr. (It is gratifying to students that this simple derivation predicts the exact value the experiment is designed to measure.) This number is approximately equal to the angle subtended by a human hair when viewed from about 3 meters away. (The verification of this analogy makes a good student exercise, as does constructing similar analogies with familiar objects.) The technology developed to do this experiment is so remarkable that the precession is expected to be measured with an uncertainty of better than 0.5 milli-arcsecond $= 1.39 \times 10^{-7}°$, or to within 0.01%.

In April, 2007 it was announced that a preliminary analysis of the data had confirmed the net precession predicted to occur from the Thomas and geodetic effects to within an experimental uncertainty of 1%.[13]

The history of Gravity Probe B and the technology developed to make such accurate measurements is a fascinating story, any part of which can easily be researched and presented as a student project. For example, the quartz gyroscopes used are the most spherical objects ever made, with radii constant to within 3×10^{-7} of an inch. They are homogeneous to within 2 ppm and coated with a layer of niobium that is 1270 nm thick. The niobium is superconducting and the magnetic moment generated by the rotating quartz-niobium sphere is detected by SQUIDs, which can measure magnetic fields on the order of 10^{-13} G. The gyroscopes are so free of their mountings that their spindown time is estimated at 15000 yr.[40]

B. The fine structure of hydrogen

The first confirmation of Thomas precession occurred in 1927 when it was used, along with the concept of spin, to correctly predict the spectrum of hydrogen. (A description of the historical context in which Thomas' papers appeared is given by both Uhlenbeck[12] and Tomonaga.[42]) The way Thomas precession enters the calculation is as follows: Part of the potential energy associated with the spin of the electron (the "spin-orbit coupling") is first calculated in the rest frame of the electron and then transformed back to the rest frame of the proton. However, because it was calculated in the electron's frame, the potential energy associated with the electron's spin vector includes the energy associated with the Thomas precession and this energy must be subtracted off in order to obtain the correct potential energy in the proton's frame. In the early 1920s, before Thomas pression was known, physicists were puzzled because the predicted spectrum didn't agree with what was found experimentally. After Thomas precession was included in the calculation the predicted spectrum agreed with what was found experimentally, not only for the hydrogen atom, but also for hydrogen-like atoms, alkali atoms, etc. As Tomonaga remarked, "Thus all the fog of 1923-24 [was] completely cleared."[42]

VIII. CONCLUSION

We have presented two examples which can be used in relativity courses for majors and nonmajors. Both involve circular motion, and both make predictions which are verified in experiments with macroscopic objects. The first, called the circular twin paradox, has been confirmed with atomic clocks flown in airplanes. The second, Thomas precession, has been confirmed in the gyroscope experiment on the satellite Gravity Probe B.[13]

Both of these examples can be discussed in a relativity course after time dilation and length contraction have been derived. Their confirmation with macroscopic objects helps convince students of the validity of relativity, and shows them that its seemingly bizarre predictions about space and time must be taken seriously.

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APPENDIX A: THE ACCELERATION OF THE MUONS IN THE EXPERIMENTS AT CERN

If students are familiar with the Lorentz transformations for the velocity and acceleration, then calculating the acceleration experienced by the muons in the experiments of Bailey *et al.*[2] makes a simple and interesting application that can be done in class or assigned as a problem.

In Ref. 2 the muons were moving in a circular orbit of radius 7.00 m with a constant speed of $v=0.99942$ c ($\gamma=29.304$). In this case, the relativistic form of Newton's second law is

$$\vec{\mathbf{F}} = \frac{d(\gamma m \vec{\mathbf{v}})}{dt} = \gamma m \vec{\mathbf{a}}. \tag{A1}$$

Because the force holding the muons in orbit is magnetic, $qvB=\gamma ma$ and

$$a = \frac{qvB}{\gamma m}. \tag{A2}$$

Using the value of $B=1.472$ T reported by Bailey *et al.*,[2] and expressing the mass of the muon as 206.7 times the mass of the electron, Eq. (A2) tells us that $a=1.28\times 10^{16}\ \mathrm{m/s^2}$. However, this is the acceleration measured in the lab frame. The acceleration experienced by the muons is the proper acceleration which, for circular motion, is $\gamma^2\ a$.[43] Consequently, the muons experience an acceleration of $1.1 \times 10^{19}\ \mathrm{m/s^2}$, or approximately 10^{18} times the acceleration of gravity, which is the value given in Ref. 2.

APPENDIX B: HOW THE TRAVELING TWIN EXPLAINS WHY HER CLOCK RUNS SLOW RELATIVE TO THE EARTH TWIN

In Sec. IV we described the twin, triplet, and circular twin paradoxes from the point of view of the Earth observer. Students often ask how the traveler describes the trip. In this section we answer this question in the context of special relativity by elaborating on the approach used by Muller.[44]

To introduce some notation, consider again the standard twin paradox from the point of view of the Earth observer. Suppose the Earth observer and the traveler agree that the traveler's clock begins at the location of the Earth observer's clock with $x'=x=0$ when $t=t'=0$. Also suppose the Earth twin observes the traveler move to the right until she reaches the turning point $x=D$, when the Earth clock reads $T=D/v$. If we use the Lorentz transformation, we find that if the traveler's clock says this event occurs at a time T_T, then

$$T_T = \gamma\left(T - \frac{vD}{c^2}\right) = \gamma\left(\frac{D}{v} - \frac{vD}{c^2}\right) = \frac{D}{v\gamma}, \tag{B1}$$

which implies that

$$T_T = \frac{T}{\gamma}. \tag{B2}$$

Hence the Earth observer says the traveler's clock will read $T_T=T/\gamma$ and, thus, that the traveler's clock is running slow by a factor of γ. Because the Earth observer knows that acceleration doesn't affect the rate of an ideal clock, he concludes that when the traveler returns to the origin, the Earth clock will read $2D/v$ and the traveler's clock will read $2D/(\gamma v)$.

As an example, suppose $v=0.6$ c and $D=3$ c-years (that is, 3 light-years). Then according to the Earth observer, the turning point is reached at time

$$T = \frac{D}{v} = \frac{3\ \text{c-years}}{0.6\ \text{c}} = 5\ \text{yr}. \tag{B3}$$

The Earth observer predicts that when the traveler reaches the turning point her clock will read

$$T_T = \frac{D}{\gamma v} = \frac{3\ \text{c-years}}{1.25 \times 0.6\ \text{c}} = 4\ \text{yr}. \tag{B4}$$

Consequently, the Earth observer concludes that when the traveling clock returns to the origin and the two clocks are compared, the Earth clock will read

$$\frac{2D}{v} = 10\ \text{yr}, \tag{B5}$$

and the traveler's clock will read

$$\frac{2D}{\gamma v} = 8\ \text{yr}. \tag{B6}$$

Now let's consider what happens from the traveler's point of view. The traveler observes the Earth clock moving to the left until it reaches its turning point at $x_T=-vT_T$. By using the Lorentz transformation for a frame moving with velocity $-v$, the traveler predicts that when the Earth clock arrives at the turning point it will read

$$T = \gamma\left(T_T - \frac{(-v)\cdot x_T}{c^2}\right) = \gamma\left(T_T - \frac{v^2 T_T}{c^2}\right) = \frac{T_T}{\gamma}. \tag{B7}$$

Hence the traveler says the clock held by the Earth observer is running slow by a factor of γ, and we have the makings of a paradox.

Let's continue to examine the situation from the traveler's frame. If the distance to the turning point in the Earth frame is $D=3$ c-years, then according to the traveler, that distance is $D_T=D/\gamma=(3\ \text{c-years}/1.25)=2.4$ c-years. Consequently, when the Earth clock reaches the turning point, the traveler's clock reads $T_T=D_T/v=(2.4\ \text{c-years}/0.6\ \text{c})=4$ yr (as we expected from Eq. (B4)). In the traveler's frame the turning point is at $x_T=-D_T$ so, according to the traveler, when the Earth clock reaches this location, it will read

$$T = \gamma\left(T_T - \frac{(-v)\cdot(-D_T)}{c^2}\right) \tag{B8}$$

$$= 1.25\left(4\ \text{yr} - \frac{(-0.6\ \text{c})\cdot(-2.4\ \text{c-years})}{c^2}\right) \tag{B9}$$

$$\Rightarrow T = 3.2\ \text{yr}. \tag{B10}$$

This answer is just what the traveler expects because, in their frame, it is the Earth clock that is moving and so must be running slow by a factor of γ, as we showed in Eq. (B7).

Now the traveler changes into a frame that is returning to the Earth. In this frame the traveler observes the Earth clock traveling toward her with a speed $v=0.6$ c. We can calculate the reading on the Earth clock measured by the traveler just after the change in two ways. First, if the change is quick, then the time T_T on the traveler's clock just after the change is essentially the same as the time on the traveler's clock just before the change. Hence, just after the change, the traveler can use the standard Lorentz transformation to show that the Earth clock now reads

$$T=\gamma\left(T_T-\frac{vD_T}{c^2}\right), \tag{B11}$$

$$=1.25\left(4\ \text{yr}-\frac{0.6\ \text{c}(-2.4\ \text{c-years})}{c^2}\right)=6.8\ \text{yr}. \tag{B12}$$

Consequently the traveler measures the Earth clock to read one value (3.2 yr) just before it changes into an inertial frame traveling back to the Earth and a different value (6.8 yr) just after.

Now consider a second way to model the turnaround. If the traveler is moving in a charged spacecraft, then the turnaround can occur by having the spacecraft enter a constant magnetic field perpendicular to the plane of its motion. In this case the speed of the spacecraft is constant as it moves through a semicircle and then heads back to the Earth. Both the size of the semicircle and the time it takes for the spacecraft to traverse it can be made small by increasing the strength of the magnetic field.

Modeling the turnaround in this way provides a better understanding of what happens as it occurs. The general equation for the time transformation is

$$T=\gamma\left(T_T-\frac{\vec{\mathbf{v}}\cdot\vec{\mathbf{x}}_T}{c^2}\right), \tag{B13}$$

where $\vec{\mathbf{v}}\cdot\vec{\mathbf{x}}=vx\cos\theta$. Because the Earth observer is moving in the $-\hat{\mathbf{i}}$ direction, and because $\vec{\mathbf{x}}_T$ points in the $-\hat{\mathbf{i}}$ direction, the traveler observes $\theta=0$ just before the turn-around and $\theta=\pi$ just after.

Because the turnaround is caused by a magnetic field, $\theta=\omega t=(qB/m)t$ and Eq. (B13) says the traveler will observer a nonlinear speed-up of the Earth twin's clock. The traveler thus observes clocks, and all processes in the Earth frame—including the aging of the Earth twin—to speed up during the turnaround. But did the Earth twin really age more quickly during the turnaround? There is an absolute answer to this question because the Earth twin can be attached to machines that measure the aging process in terms of heart rate, metabolic rate, etc. A digital display of the pulse, for example, would not show a speed-up of the Earth twin's heartbeat during the turnaround. In fact, we can understand the aging process of the Earth twin as observed from the traveler's frame in the same way we understood the acceleration of the Earth twin as observed from the traveler's frame: Just as the accelerometer in the Earth twin's frame doesn't register an acceleration, a heart monitor attached to the Earth twin would not register a faster heartbeat. Thus, even though the traveling twin observes the Earth twin to accelerate and his clocks to run fast during the turnaround, the Earth twin doesn't really accelerate as measured by an accelerometer and doesn't really age more rapidly as measured by a heart or metabolic monitor.

We can now summarize the complete trip from the traveler's point of view. Initially the traveler observes the Earth clock moving away from her with a speed $v=0.6$ c. Just as the clock reaches the turning point, the traveler observes that 4 yr have passed on her clock but only 3.2 yr have passed on the Earth clock. This time difference is what the traveler expects because the Earth clock is moving in her frame and so is running slow by a factor of γ. During the turnaround, the traveler observes the Earth clock speed-up. At the end of the turnaround, when the Earth clock is just beginning its trip back to the traveling twin, the traveler says the Earth clock's reading has changed from 3.2 to 6.8 yr. During the return trip, the traveler again observes that 4 yr pass on her clock while only 3.2 yr pass on the Earth clock. Consequently, when the Earth clock and the traveler's clock meet, the traveler expects the Earth clock to read $(6.8+3.2)$ yr=10 yr. So the traveler completely understands why the round trip registered 8 yr on her clock and 10 yr on the Earth clock.

What happens during the turnaround can also be understood from the relativity of simultaneity. As Mermin[45] has pointed out, just before the turnaround, the traveler observes what the Earth clock reads "now" from a frame in which the Earth clock is traveling away, while just after the turnaround the traveler observes what the Earth clock reads "now" from a frame in which the Earth clock is returning. But two events separated in space that are simultaneous in one frame are not necessarily simultaneous in another. If two events are simultaneous in the Earth frame, then after applying the Lorentz transformation Eq. (B9), they are not simultaneous in the traveler's frame and the time interval between them is $\gamma vD/c^2$, where D is the separation of the two events in the Earth frame. The key point is that in Eq. (B9) the velocity is negative and in Eq. (B11) it is positive. Therefore, when we calculate the difference in the reading on the Earth clock before and after the traveler changes frames, we find it is $2\gamma vD/c^2=2(1.8\ \text{c-years})=3.6$ c-years, which is what we found in Eqs. (B10) and (B12).

Another way to understand what happens during the turnaround is to note that when observed from the traveler's frame the clocks in the Earth frame are not synchronized correctly. By observing clocks being synchronized in a frame moving away from her, the traveler finds that each successive clock is set out of sync with the clock at the origin by a factor of $\gamma vx/c^2$, where x is the proper distance between the clock being synchronized and the clock at the origin. When the traveler is in a frame in which the Earth is traveling away in the negative $-\hat{\mathbf{i}}$ direction, she observes that the reading on the Earth clock at the turnaround point is behind what it would have been if the Earth clocks had been synchronized correctly. (The trailing event—the reading on the Earth clock at the turnaround point in this case—occurs first.) In a frame in which the Earth clock is approaching, the traveler observes that the reading on an Earth clock at the turnaround point is ahead of what it would have been if the Earth clocks had been synchronized correctly. These two effects add when the traveler changes frames and hence the traveler observes the reading on an Earth clock to advance during the turnaround by $2\gamma vD/c^2=2(1.8\ \text{c-years})=3.6$ c-years from 3.2 to 6.8 yr.

The same three analyses can be extended to a twin who travels around a polygon. For example, suppose the Earth observer has set up a system of synchronized clocks in his frame. When the traveling twin accelerates at each vertex of the polygon, she will see the Earth clock at that vertex advance more quickly during the acceleration. The advances accumulate so that when the traveling twin is again next to the Earth twin, both will agree the clock carried around the polygon has recorded less time than the one held by the Earth twin. The same thing happens in the limit as the polygon path becomes a circular path.

APPENDIX C: STANDARD FORM OF THE EQUATION FOR THOMAS PRECESSION

In circular motion with constant speed the acceleration $\vec{\mathbf{a}} = (-v^2/r)(\hat{\mathbf{r}})$ and the linear velocity $\vec{\mathbf{v}} = \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}$, where $\vec{\boldsymbol{\omega}} = (2\pi/P)\hat{\mathbf{k}}$ and P is the period. Consequently,

$$\vec{\mathbf{a}} \times \vec{\mathbf{v}} = \vec{\mathbf{a}} \times (\vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}) = \vec{\boldsymbol{\omega}}(\vec{\mathbf{a}} \cdot \vec{\mathbf{r}}) = -v^2 \vec{\boldsymbol{\omega}}. \tag{C1}$$

By using this result in Eq. (1) we find that Thomas precession of a gyroscope can be expressed in vector form as

$$\vec{\boldsymbol{\omega}}_T = (\gamma - 1)\frac{\vec{\mathbf{a}} \times \vec{\mathbf{v}}}{v^2}, \tag{C2}$$

where $\vec{\mathbf{a}}$ is the acceleration of the traveling twin and $\vec{\mathbf{v}}$ is her velocity. Equation (C2) is the exact (and standard) form of the Thomas precession. If v/c is small, $\gamma \approx 1 + v^2/(2c^2)$, and

$$\vec{\boldsymbol{\omega}}_T = \frac{\vec{\mathbf{a}} \times \vec{\mathbf{v}}}{2c^2} + O\left(\frac{v^3}{c^4}\right). \tag{C3}$$

We can use Eq. (C3) to derive the magnitude and direction of the Thomas precession predicted to occur in GP-B. Because the satellite is held in orbit by the gravitational force, the acceleration of its reference frame is $\vec{\mathbf{a}} = -(GM/r^2)\hat{\mathbf{r}}$, where r is the radius of the orbit and M is the mass of the Earth. In this case, because $a = v^2/r$, $\vec{\mathbf{v}} = (\sqrt{GM/r})\hat{\mathbf{v}}$. Consequently, a gyroscope in the satellite will be observed by the traveling twin to precess with an angular speed

$$\omega_T = \frac{1}{2c^2}\frac{GM}{r^2}\sqrt{\frac{GM}{r}} = \frac{1}{2c^2 r}\left(\frac{GM}{r}\right)^{3/2} \tag{C4}$$

in the direction

$$\hat{\boldsymbol{\omega}}_T = -\hat{\mathbf{r}} \times (\hat{\mathbf{v}}), \tag{C5}$$

which is antiparallel to the angular momentum of the satellite in its orbit. Thus, relative to the traveling twin, the gyroscope will precess in the opposite sense of the orbit of the satellite. That is, if the satellite is orbiting the Earth in the counterclockwise direction, the traveler will observe the gyroscope to precess in the plane of the orbit in the clockwise direction.

The same analysis can be used to understand Thomas precession in the hydrogen atom. In this case the "satellite" is the electron and the "gyroscope" is the electron's spin vector. The acceleration in Eq. (C2) is $\vec{\mathbf{a}} = -(ke^2/m_e r^2)\hat{\mathbf{r}}$, where k is a constant, m_e is the mass of the electron, and e is the charge of the electron. If we replace GM in Eq. (C4) by ke^2/m_e, the Thomas precession of the electron's spin vector in its rest frame is

$$\omega_T = \frac{1}{2c^2 r}\left(\frac{ke^2}{m_e r}\right)^{3/2}, \tag{C6}$$

and the direction of $\vec{\boldsymbol{\omega}}_T$ is opposite to that of the electron's orbital angular momentum.

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[1]As is usual in discussions of the twin paradox at the introductory level, we pretend the Earth is an inertial frame and ignore its spin and motion around the Sun. Although the paradox should be discussed with one of the twins in an inertial frame rather than on Earth, it loses some of its dramatic appeal when presented in this way. We discuss how the Earth's motion is taken into account when we consider the experiments of Hafele and Keating. (Ref. 4)

[2]J. Bailey *et al.*, "Measurements of relativistic time dilatation for positive and negative muons in a circular orbit," Nature (London) **268**, 301–305 (1977); "Final report on the CERN muon storage ring including the anomalous magnetic moment and the electric dipole moment of the muon, and a direct test of relativistic time dilation," Nucl. Phys. B **150**, 1–75 (1979).

[3]J. J. Hay, J. P. Schiffer, T. E. Cranshaw, and P. A. Egelstaff, "Measurement of the red shift in an accelerated system using the Mössbauer effect in Fe^{57}," Phys. Rev. Lett. **4**, 165–166 (1960).

[4]J. C. Hafele and R. E. Keating, "Around-the-world atomic clocks: Predicted relativistic time gains," Science **177**, 166–168 (1972); and **177**, 168–170 (1972); J. C. Hafele, "Relativistic time for terrestrial circumnavigations," Am. J. Phys. **40**, 81–85 (1972).

[5]C. O. Alley, "Proper time experiments in gravitational fields with atomic clocks, aircraft, and laser light pulses," in *Quantum Optics, Experimental Gravitation, and Measurement Theory*, edited by P. Meystre and M. O. Scully (Plenum, New York, 1983), pp. 347–363; W. Sullivan, "Einstein and clocks," Appl. Opt. **17**, 5 (1978).

[6]A. P. Lightman, W. H. Press, R. H. Price, and S. A. Teukolsky, *Problem Book in Relativity and Gravitation* (Princeton U.P., Princeton, NJ, 1975) pp. 6,7, 138–140.

[7]M. B. Cranor, E. M. Heider, and R. H. Price, "A circular twin paradox," Am. J. Phys. **68**, 1016–1020 (2000).

[8]See Ref. 7, p. 1016.

[9]L. H. Thomas, "Motion of the spinning electron," Nature (London) **117**, 514 (1926); "The kinematics of an electron with an axis," Philos. Mag. **3**, 1–23 (1927). The first article is a letter announcing Thomas' result and the second presents a complete discussion of it. For a more complete set of references see Ref. 32.

[10]J. R. Taylor, *Classical Mechanics* (University Science Books, Sausalito, CA, 2005), p. 670, Problem 15.28.

[11]J. P. Costella, B. H. J. McKellar, and A. A. Rawlinson, "The Thomas rotation," Am. J. Phys. **69**, 837–847 (2001).

[12]G. E. Uhlenbeck, "Personal Reminiscences," Phys. Today **29**(6), 43–48 (1976).

[13]See ⟨http://einstein.stanford.edu/⟩and ⟨http://einstein.stanford.edu/content/press_releases/SU/pr-aps-041807.pdf⟩.

[14]N. David Mermin, *It's About Time* (Princeton U.P., Princeton, NJ, 2005).

[15]R. Baierlein, *Newton to Einstein: The Trail of Light* (Cambridge U.P., Cambridge, 2005).

[16]For a quantitative discussion of the standard twin paradox see Appendix B.

[17]A surface on which the ratio of the circumference of a circle to its radius is less than 2π has positive curvature, like the surface of a sphere. The Earth provides a simple example of this situation that is easy to visualize. Imagine we are looking down on the Earth from the North Pole. To us, a circle formed by a latitude line above the equator has a radius equal to the length of the longitude line from the North Pole to the circle, which is $R\theta$, where R is the radius of the Earth and θ is the angle the latitude line makes with the z axis. In contrast, the actual radius of the circle is $R \sin\theta$. Consequently, to an observer looking down from the North Pole, the ratio of the circumference of a latitude line to its radius is $2\pi R \sin\theta/(R\theta) < 2\pi$.

[18]A. Einstein, *The Meaning of Relativity* (Princeton U.P., Princeton, NJ, 1966), pp. 59–61; *Relativity: The Special and the General Theory* (Three Rivers Press, New York, 1961), pp. 88–91.

[19]This is the experimental precision given in Y. Z. Zhang, *Special Relativity and its Experimental Foundations* (World Scientific, Singapore, 1997), p. 194.

[20]This experimental precision is given in W. Rindler, *Relativity: Special, General, and Cosmological* (Oxford U.P., Oxford, 2006), 2nd ed., p. 67.

[21]Reference 19, pp. 180–183, and Ref. 22, pp. 95–98 also discuss this point.

[22]H. C. Ohanian, *Special Relativity: A Modern Introduction* (Physics Curriculum and Instruction, Lakeville, MN, 2001).

[23] This graph is in Ref. 5, Fig. 47.

[24] Just as the famous experiment with muons created in the upper atmosphere is shown in a film (Ref. 25), so too are film clips of the experiment of Alley et al. (Ref. 5) included in the BBC documentary "Einstein's universe," which can be obtained from Corinth Films (Ref. 26). Showing both films in class not only provides a welcome change of pace but also reinforces the reality of time dilation.

[25] D. H. Frisch and J. H. Smith, "Time dilation–An experiment with mu-mesons," Education Development Center, Newton, MA, 1963. See also Am. J. Phys. **31**, 342–355 (1963).

[26] "Einstein's universe," produced by the BBC and WGBH, 1979. Available from Corinth Films, 3117 Bursonville Rd., Riegelsville, PA 19077, ⟨www.store.corinthfilms.com⟩.

[27] This precision is quoted in Ref. 28, p. 1357.

[28] D. Newman, G. W. Ford, A. Rich, and E. Sweetman, Phys. Rev. Lett. **40**, 1455–1358 (1978).

[29] In class we define a gyroscope as simply a mass spinning around an axis. We say that the key to the gyroscope is that its mounting can move in any direction without exerting a torque back on the axis about which the mass is spinning. To the degree that no torques act, angular momentum is conserved and the gyroscope points in a fixed direction no matter how its mounting moves.

[30] This derivation is based on the discussion presented in the appendix of Ref. 31, which the author says is based on an idea of Purcell. The derivation is especially attractive because it only requires knowledge of length contraction, whereas most derivations of Thomas precession are much more complicated (see, for example, the references in Rhodes and Semon—Ref. 32).

[31] Richard A. Muller, "Thomas precession: Where is the torque?," Am. J. Phys. **60**, 313–317 (1992).

[32] John A. Rhodes and Mark. D. Semon, "Relativistic velocity space, Wigner rotation and Thomas precession," Am. J. Phys. **72**, 943–960 (2004).

[33] R. L. Reese, *University Physics for Scientists and Engineers* (Brooks-Cole, New York, 2000), pp. 138–140.

[34] M. L. Boas, *Mathematical Methods in the Physical Sciences* (Wiley, New York, 2005), pp. 283–288.

[35] G. Pugh, "Proposal for a satellite of the coriolis prediction of general relativity," 1959 Pentagon Weapons Evaluation Group (WSEG), memo one. This paper is reprinted in Ref. 37, pp. 414–426, and is available at ⟨http://einstein.stanford.edu/content/sci_papers/papers/Pugh_G_1959_109.pdf⟩.

[36] L. I. Schiff, "Possible new experimental test of general relativity theory," Phys. Rev. Lett. **4**, 215–217 (1960); L. I. Schiff, "Motion of a gyroscope according to Einstein's general theory of relativity," Proc. Natl. Acad. Sci. U.S.A. **46**, 871–882 (1960). This paper is reprinted in Ref. 37, pp. 427–438 and at ⟨http://einstein.stanford.edu/content/sci_papers/papers/Schiff_LI_1960_110.pdf⟩.

[37] *Nonlinear Gravitodynamics: The Lense-Thirring Effect*, edited by R. Ruffini and C. Sigismondi (World Scientific, Singapore, 2003).

[38] J. Lense and H. Thirring, "Uber den einfluss Der Eigenrotation der Zentral korper auf die Bewegung der Planeten und Monde nach der Einsteinschen Gravitationstheorie," Phys. Z. **19**, 156–163 (1918). See also Ref. 37, pp. 349–388, for an English translation of this paper.

[39] B. R. Holstein, "Gyroscope precession and general relativity," Am. J. Phys. **69**, 1248–1256 (2001).

[40] Excellent discussions of the history, technology, and physics of Gravity Probe B can be found at ⟨http://einstein.stanford.edu⟩, ⟨www.nasa.gov/pdf/168808main_gp-b_pfar_cvr-pref-execsum.pdf⟩, ⟨http://books.nap.edu:80/html/gpb/summary.html⟩, and ⟨http://einstein.stanford.edu/highlights/GP-B_Launch_Companion.pdf⟩.

[41] The actual orbit of the satellite has a perigee altitude of 639.5 km above the Earth and an apogee altitude of 659.1 km. We are using the average value 649.3 km. Note that Francis Everitt, in his talk "Testing Einstein in Space: The Gravity Probe B Mission," Stanford University, May 18, 2006 (⟨http://einstein.stanford.edu/ ⟩), used the value 642 km.

[42] S. Tomonaga, *The Story of Spin* (University of Chicago Press, Chicago, 1974), Chaps. 2 and 11. The quote at the end of this paragraph is on pp. 41–42.

[43] Not all texts calculate the Lorentz transformation equations for acceleration. One that does is Rindler (Ref. 20), pp. 70–71, 75.

[44] Richard A. Muller, "The twin paradox in special relativity," Am. J. Phys. **40**, 966–969 (1978).

[45] Reference 14, pp. 119–123, 177, and 178.