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The Aharonov–Bohm Effect: Still a Thought-Provoking Experiment

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In the Aharonov–Bohm effect, electromagnetic potentials alter the two-slit interference pattern formed by an electron beam. We discuss here a curious feature of this effect, namely that, even though the interference pattern changes, none of its moments are shifted.

1. INTRODUCTION

Nearly 30 years have passed since Aharonov and Bohm suggested the beautiful effect that bears their name.(1) Theoretically, their paper led to a substantial reassessment of the role of electromagnetic potentials in quantum physics.(2) Experimentally, it inspired a whole series of elegant, landmark experiments.(3)

Over the years the Aharonov–Bohm (or AB) effect has continued to stimulate new insights and ideas. In particular, in two recent papers(4,5) we have drawn attention to a curious feature of the AB effect that had attracted relatively little notice. This feature, which is interesting both physically and mathematically, is easily described: The AB experiment is a two-slit experiment with electrons. A magnetic or electric potential is introduced and causes the interference pattern to change. The result we presented in Refs. 4 and 5 is that, even though the two-slit pattern changes, nevertheless none of the moments of position or momentum are shifted. Our main purpose in Refs. 4 and 5 was to give two different proofs of this result, which we called the no-shift theorem. In this paper we shall not repeat the details of those proofs. Rather, we wish to describe *why* the result is true, to give

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some simple examples that illustrate it, and to discuss the seeming paradox that two different distributions can have all of their corresponding moments equal.

We offer this work in the Bohm *Festschrift* as a tribute to the continuing resilience and fruitfulness of the ideas contained in the original paper of Aharonov and Bohm.

2. A SIMPLE EXAMPLE

For our present purposes the important feature of the AB effect is that a two-slit interference pattern is altered by shifting the relative phase of the two transmitted waves. In the AB experiment the wave in question is a quantum wave, and the phase shift is caused by a magnetic or electric potential. However, a similar shift could be produced in several classical waves; for example, with water waves a change in the depth near one slit would shift the phase of the wave passing through that slit; with light waves, a change in refractive index near one slit would produce the same effect.

To illustrate the result of a phase shift in one of the transmitted waves, we use the familiar model of Frauenhoffer diffraction from two infinitely long, rectangular slits, whose width we denote by $2a$ and center-to-center separation by b. In the approximation that $\lambda \ll a$, the transmitted intensity (in the absence of the AB phase shift) is well known to be

$$I_0(x) \propto [1 + \cos kbx] \left(\frac{\sin kax}{kax}\right)^2 \tag{1}$$

Here k is the incident wave number ($k = 2\pi/\lambda$) and x is the distance from the center of the observation screen, measured in the direction perpendicular to the slits. The first factor, $[1 + \cos kbx]$, in Eq. (1) describes the two-slit interference. The second factor is the well-known "sinc" function, which gives the intensity transmitted by either slit separately. In Fig. 1 this "sinc" function is shown as a solid curve, and the intensity $I_0(x)$, resulting from the two-slit oscillations underneath the one-slit envelope, is shown as the dashed curve.

If we introduce a phase shift α between the two transmitted waves, then the intensity $I_0(x)$ of Eq. (1) is changed to

$$I_\alpha(x) \propto [1 + \cos(kbx - \alpha)] \left(\frac{\sin kax}{kax}\right)^2 \tag{2}$$

Comparing (2) and (1) we see that the result of the phase shift α is to move the two-slit pattern sideways underneath the unaltered one-slit envelope. This is illustrated for the case $\alpha = \pi/3$ by the dotted curve in Fig. 1.

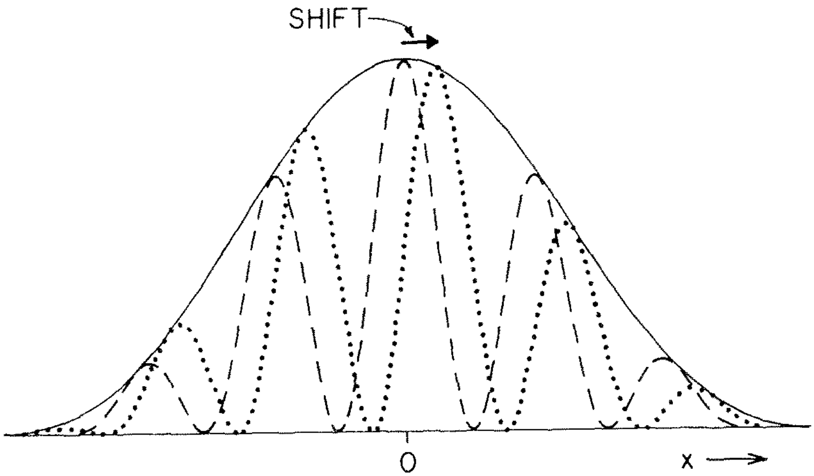

Fig. 1. Two-slit intensity for two infinitely-long rectangular slits. The solid curve is the one-slit envelope. The dashed curve is the interference pattern observed when there is no Aharonov–Bohm phase shift ($\alpha = 0$), and the dotted curve shows the effect of a phase shift $\alpha = \pi/3$.

The no-shift theorem implies that although the distributions $I_0(x)$ and $I_\alpha(x)$ are certainly different (except when α is a multiple of 2π), corresponding moments of the two are all the same. That is, for all $n = 0, 1, 2, \ldots$ the moment

$$\langle x^n \rangle_\alpha = \int x^n I_\alpha(x)\, dx \tag{3}$$

is independent of α. Perhaps the most surprising aspect of this result is that it contradicts one's intuition that any "reasonable" distribution should be determined uniquely by its moments; that is, if all the corresponding moments of two distributions are equal, then the distributions themselves should be the same. We shall return to this point in Section 4. For now we remark only that the two-slit interference patterns are clearly not "reasonable" in this sense. Here we wish to examine the moments $\langle x^n \rangle_\alpha$ for the distribution (2) of two parallel slits and to see explicitly that they are independent of α.

The zeroth moment $\langle x^0 \rangle_\alpha$ is simply the integrated intensity transmitted through the slits, and one would not *expect* it to vary with the phase shift α. A straightforward calculation shows that this is correct:

$$\langle x^0 \rangle_\alpha - \langle x^0 \rangle_0 = \int [I_\alpha(x) - I_0(x)]\, dx$$

$$\propto (\cos\alpha - 1) \int_{-\infty}^{\infty} \cos(kbx) \left[\frac{\sin kax}{kax}\right]^2 dx \tag{4}$$

This integral can be evaluated by contour integration [or see, for example, Gradshteyn and Ryzhik,[6] p. 450, Eq. (5)] and is zero provided $b > 2a$. However, the condition $b > 2a$ is simply the condition that the two slits do not overlap. Thus the shift (4) in the zeroth moment is zero under precisely the conditions of the experiment.

If we look next at the first moment $\langle x \rangle_\alpha$, the no-shift theorem is, at first glance, quite surprising. It is clear from Fig. 1 that the effect of the phase shift α is to push the whole distribution to the right (for $\alpha > 0$). This suggests that the centroid $\langle x \rangle_\alpha$ of the distribution must inevitably increase. However, closer inspection makes clear that this conclusion is not necessarily correct. The shift of the peaks in Fig. 1 occurs *underneath the fixed one-slit envelope*. Thus the individual peaks also change height as they shift sideways. To first order in α, the height of the central peak is unchanged; but the $n = 1$ peak becomes *lower*, while the $n = -1$ peak becomes *higher*. Since the $n = 1$ peak has $x > 0$, while the $n = -1$ peak has $x < 0$, this effect tends to change $\langle x \rangle$ in the *negative* direction. Therefore, it is at least conceivable that the tendency for $\langle x \rangle$ to *increase* because of the sideways shift of the pattern is exactly cancelled by this tendency for $\langle x \rangle$ to *decrease* because of the changing heights of the individual maxima.

A straightforward calculation shows that the cancellation just described does indeed occur:

$$\langle x \rangle_\alpha - \langle x \rangle_0 = \int_{-\infty}^{\infty} [I_\alpha(x) - I_0(x)]\, x\, dx$$

$$\propto \sin \alpha \int_{-\infty}^{\infty} \sin(kbx) \left[\frac{\sin kax}{kax}\right]^2 x\, dx \qquad (5)$$

Like the integral of Eq. (4), this integral can be evaluated [or see Gradshteyn and Ryzhik,[6] p. 450, Eq. (3)] and it is zero provided $b > 2a$ (that is, provided the slits do not overlap).

We see that, for the interference pattern (2) of two parallel slits, the first two moments, $\langle x^0 \rangle_\alpha$ and $\langle x^1 \rangle_\alpha$, are indeed independent of α, as predicted by the no-shift theorem. If we go on to look at the second moment, $\langle x^2 \rangle_\alpha$, we find a different surprise: The second and higher moments are all divergent! We shall discuss the reason for these divergences in Section 4. Here we make just two comments: First, the divergence is not really surprising since the distribution (2) is the result of our model, which involves an infinite plane wave incident on the two-slit barrier. Second, the divergence does not actually contradict the no-shift theorem, since the latter states only that $\langle x^n \rangle_\alpha$, *if finite*, is independent of α.

3. PROVING THE NO-SHIFT THEOREM

For detailed proofs of the no-shift theorem for the moments of position and momentum, we refer the reader to our previous two papers.[4,5] Here we sketch one proof for the case of position, because it is easily followed and illustrates clearly the role of the various assumptions we make.

Following Ref. 4 we assume that, in the absence of the AB phase shift, the transmitted wave packet has the form

$$\psi_0(\mathbf{r}, t) = \psi_1(\mathbf{r}, t) + \psi_2(\mathbf{r}, t) \tag{6}$$

where ψ_ν ($\nu = 1$ or 2) is the packet that would be transmitted through the νth slit alone. We assume that at some time, $t = 0$, just after the wave passes the barrier, the two packets $\psi_1(\mathbf{r}, 0)$ and $\psi_2(\mathbf{r}, 0)$ do not overlap, and further that for times $t > 0$ their motion is unaffected by the barrier; that is, for $t > 0$, $\psi_\nu(\mathbf{r}, t)$ evolves like a free wave packet. The effect of the AB phase shift α is to replace the wave $\psi_0(\mathbf{r}, t)$ in (6) by[3]

$$\psi_\alpha(\mathbf{r}, t) = \psi_1(\mathbf{r}, t) + e^{i\alpha}\psi_2(\mathbf{r}, t) \tag{7}$$

The moments of the transmitted wave ψ_α are just the expectation values of powers of x, y, and z, and products of these. For simplicity here, we consider a simple power of x. (Detailed proofs for all moments can be found in Ref. 4.) We consider therefore the expectation value

$$\langle x^n \rangle_\alpha = \langle \psi_\alpha(\mathbf{r}, t), x^n \psi_\alpha(\mathbf{r}, t) \rangle \tag{8}$$

Substituting (7) into (8), we obtain

$$\langle x^n \rangle_\alpha = \langle \psi_1, x^n \psi_1 \rangle + \langle \psi_2, x^n \psi_2 \rangle + 2 \operatorname{Re} e^{i\alpha} \langle \psi_1(\mathbf{r}, t), x^n \psi_2(\mathbf{r}, t) \rangle \tag{9}$$

All three terms here depend on t (though we have only shown this explicitly with the last), but only the third term depends on the phase shift α. However, this third term is in fact zero, as we now show: Since the two

[3] This is a slight oversimplification. In the magnetic AB effect, for example, each wave picks up an $\mathbf{r}$-dependent phase factor. However, the relative phase is independent of $\mathbf{r}$ and this complication does not affect the present argument at all.

wave packets ψ_1 and ψ_2 evolve freely for $t>0$, we can rewrite the matrix element in the third term as

$$\langle \psi_1(\mathbf{r}, t), x^n \psi_2(\mathbf{r}, t)\rangle = \left\langle \psi_1(\mathbf{r}, 0), \left(x + \frac{p_x t}{m}\right)^n \psi_2(\mathbf{r}, 0)\right\rangle$$
$$= \left\langle \psi_1(\mathbf{r}, 0), \left(x - \frac{it}{m}\frac{\partial}{\partial x}\right)^n \psi_2(\mathbf{r}, 0)\right\rangle$$

Since the two packets $\psi_1(\mathbf{r}, 0)$ and $\psi_2(\mathbf{r}, 0)$ do not overlap, this matrix element is zero. This means that $\langle x^n \rangle_\alpha$ in (9) contains only two terms, neither of which depends on α, and our proof is complete.

4. A MATHEMATICAL QUESTION

We now return to the question raised in Section 2: Is it mathematically possible to have two different distributions whose corresponding moments are all equal? If the answer were "no," then our no-shift theorem would have to be false. Naturally, however, we are going to see that the answer is "yes." In fact, the two-slit distributions discussed in Section 2 give an example for which the answer *is* "yes," but since most of their moments are infinite, this example is less than completely satisfactory. Clearly the question we wish to examine is really this: Can two different distributions, all of whose moments are finite, have all corresponding moments equal? For simplicity we consider distributions $I(x)$ in a single variable.

If a "reasonable" (e.g., continuous) distribution has compact support (vanishes outside some finite interval), then indeed it is uniquely determined by its moments. To see this, note that the function can be expanded in Legendre polynomials, and that its expansion coefficients are uniquely determined by its moments. However, it is well known that a solution of the free Schrödinger equation does not retain its compact support, even if it has compact support initially. The conditions under which distributions of noncompact support are uniquely determined by their moments are fairly complicated. (See, for example, Feller.[7]) In particular, there are known families of distributions with all corresponding moments equal. One such family, given by

$$I_\alpha(x) = \begin{cases} [1 + \alpha \sin(2\pi \ln x)]\, x^{-1} e^{-(\ln x)^2/2}, & x > 0 \\ 0, & x \leqslant 0 \end{cases}$$

is discussed by Feller.[7] It is easy to check that for any α with $-1 \leqslant \alpha \leqslant 1$

this is a positive distribution all of whose moments are finite. Further, an elementary integration shows that for each $n = 0, 1, 2,\ldots$ the moment

$$\langle x^n \rangle_\alpha = \int_{-\infty}^{\infty} x^n I_\alpha(x)\, dx$$

is independent of α.

To conclude this section, we would like to mention anoher example, which we gave in Ref. 5 and which is closely related to the two-slit interference experiment.

The well-known two-slit distribution (1) is derived from the Fresnel–Kirchhoff theory, which expresses the transmitted wave as an integral over the opening of the slits. If we assume that the two-slit barrier lies in the plane $z = 0$ and the slits are translationally invariant in the y direction, we obtain a one-dimensional distribution:

$$I_\alpha(x) \propto [1 + \cos(kbx - \alpha)]\, |\psi_1(x)|^2 \tag{10}$$

where $\psi_1(x)$ is the transmitted wave for a single slit and is given by the integral

$$\psi_1(x) = \int_{-\infty}^{\infty} f(\xi)\, e^{ikx\xi}\, d\xi \tag{11}$$

Here ξ is the coordinate (in the x direction) in the slit. The integral runs only over the opening of the single slit, but we have written it as an integral from $-\infty$ to ∞ by introducing a slit transmission function $f(\xi)$.

For a simple rectangular slit, the transmission function $f(\xi)$ is a step function:

$$f(\xi) = \begin{cases} 1, & |\xi| < a \\ 0, & \text{otherwise} \end{cases}$$

This immediately makes clear why the higher moments of the two-slit distribution of Section 2 diverge. According to (11), $\psi_1(x)$ is the Fourier transform of $f(\xi)$, and if $f(\xi)$ is discontinuous, then its Fourier transform does not vanish sufficiently rapidly as $|x| \to \infty$. This also makes clear how to find an example in which all the moments are finite. If we choose $f(\xi)$ to be a function which is infinitely differentiable and which vanishes outside the interval $-a < \xi < a$, then we can integrate (11) by parts as often as we please and show that $\psi_1(x)$ approaches zero faster than any inverse power of x as $|x| \to \infty$. This in turn guarantees that all of the moments $\langle x^n \rangle_\alpha$ of $I_\alpha(x)$ in (10) are finite.

It is now a straightforward matter to check that all of the moments $\langle x^n \rangle_\alpha$ are independent of α: Some simple trigonometry gives

$$\langle x^n \rangle_\alpha - \langle x^n \rangle_0 = (\cos\alpha - 1) \int x^n \cos(kbx)\, |\psi_1(x)|^2\, dx$$

$$+ \sin\alpha \int x^n \sin(kbx)\, |\psi_1(x)|^2\, dx$$

This difference is zero for all α if and only if both integrals are zero. This in turn is true if and only if the integral

$$\Delta_n = \int x^n e^{ikbx}\, |\psi_1(x)|^2\, dx$$

is zero. Inserting (11) for $\psi_1(x)$, we can write this integral as

$$\Delta_n \propto \int dx\, x^n \int d\xi \int d\xi'\, f(\xi)\, f(\xi')\, e^{ik(b+\xi-\xi')x}$$

$$\propto \left(\frac{\partial}{\partial b}\right)^n \int d\xi \int d\xi' \int dx\, f(\xi)\, f(\xi')\, e^{ik(b+\xi-\xi')x}$$

The integral over x gives a delta function in $b+\xi-\xi'$, and we find finally

$$\Delta_n \propto \left(\frac{\partial}{\partial b}\right)^n \int d\xi\, f(\xi)\, f(\xi+b)$$

Here b is the center-to-center separation of the two slits. Therefore, since the two slits cannot overlap,

$$f(\xi)\, f(\xi+b) = 0$$

for all ξ. This means that $\Delta_n = 0$ and the no-shift theorem is confirmed.

The important points about this example are: First, the intensity (10) with ψ_1 given by (11) is a realistic physical model for the transmitted intensity from two slits, if $f(\xi)$ is the transmission function for either slit. Second, irrespective of its physical significance, the distribution $I_\alpha(x)$ in (10) is a well-defined mathematical function which is different for each α in the range $0 \leqslant \alpha < 2\pi$ and which has the same, finite moments for all α.

5. SOME DYNAMICAL CONSIDERATIONS

The main "mystery" of the AB effect is: How do the electrons interact with the source of potential? The no-shift theorem sheds some light on this question, since it proves that, whatever the interaction is, it does not affect the expectation values of the momentum, of the kinetic energy, or of any higher power of the momentum. Thus, if one wishes to describe the interaction in terms of the exchange of some dynamical variable, one must find another quantity on which to base the discussion.

Olariu and Popescu[3] suggested that a dynamical description of the AB effect could be based on the parity operator. However, almost 20 years earlier, Aharonov, Pendleton, and Peterson,[8] when considering this question, introduced a new variable, the "modular momentum," and suggested that a study of its properties might enhance our understanding of the AB interaction. In Ref. 4 we proved some properties of this operator, which is defined as

$$D = (\sin \mathbf{b} \cdot \boldsymbol{\pi})/b$$

where $\mathbf{b}$ is a vector pointing from the center of one slit to that of the other. We proved first that

$$\langle D \rangle_\alpha = -(\sin \alpha)/2b$$

which shows that $\langle D \rangle$ *is* changed by the AB interaction. Second, we verified that D obeys an equation of motion which is a generalization of the usual equation for the momentum alone. These results, together with those of Aharonov *et al.*,[8] suggest that a further study of the "modular momentum" might provide further insight into the AB interaction.

6. SUMMARY

We have discussed our no-shift theorem, which states that the interaction in the AB effect leaves all the moments of position and momentum unchanged. We have also argued that the "modular variables" of Aharonov *et al.* could shed some light on the dynamics of the AB interaction. We trust that these brief remarks bear out our claim that the original paper of Aharonov and Bohm[1] continues, 30 years later, to be a source of surprising and thought-provoking results.

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REFERENCES

1. Y. Aharonov and D. Bohm, *Phys. Rev.* **115**, 485 (1959).
2. T. T. Wu and C. N. Yang, *Phys. Rev. D* **12**, 3845 (1975).
3. S. Olariu and I. I. Popescu, *Rev. Mod. Phys.* **57**, 339 (1985).
4. M. D. Semon and J. R. Taylor, *Nuovo Cimento B* **97**, 25 (1987).
5. M. D. Semon and J. R. Taylor, *Nuovo Cimento B* **100**, 389 (1987).
6. I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products* (Academic Press, New York, 1965).
7. W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol. II, 2nd ed. (Wiley, New York, 1971), p. 227.
8. Y. Aharonov, H. Pendleton, and E. Peterson, *Int. J. Theor. Phys.* **2**, 213 (1969).