Exploring the transition from special to general relativity

Mark D. Semon[a)]
Department of Physics, Bates College, Lewiston, Maine 04240

Shimon Malin[b)] and Stephanie Wortel[b)]
Department of Physics, Colgate University, Hamilton, New York 13346

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In a previous paper we discussed two examples of circular motion which are especially useful in relativity courses because they lead to predictions verified by experiments with macroscopic objects. We analyzed these examples by considering motion around an N-gon and then taking the limit as $N \to \infty$ and the N-gon becomes a circle. In this paper we use the same approach to illustrate how generalizing special relativity to a theory that includes non-inertial frames can lead to a non-Euclidean geometry. We also derive two properties of clocks at rest in a reference frame traveling on a circular path.
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I. INTRODUCTION

One way of teaching a new subject is to present it in the way it developed historically. For example, when teaching magnetism to undergraduates, most texts choose to infer the magnetic field and the laws it obeys from basic physical phenomena rather than derive them as relativistic effects resulting from motion through electric fields. Similarly, when making the transition from special to general relativity, many texts follow the historical development of Einstein's thought as he explained it in his papers,[1] letters,[2] and books.[3–5]

Einstein's approach to generalizing special relativity was to first consider what happens in two different types of accelerating frames: those undergoing uniform linear acceleration and those rotating with a constant angular velocity. The case of uniform linear acceleration is the one most commonly discussed because it was in this context that Einstein realized a theory of accelerating frames will also be a theory of gravity, and that phenomena which occur in an accelerating frame will also occur in an inertial frame at rest in a gravitational field.

Einstein used a frame rotating with a constant angular velocity to illustrate how generalizing special relativity to a theory that includes non-inertial reference frames could involve non-Euclidean geometry. To demonstrate how a non-Euclidean geometry could emerge, Einstein considered two material disks, one at rest in an inertial frame and the other just above it rotating with a constant angular velocity about an axis perpendicular to both disks and through their centers. In order to examine the effect of rotation on the measurements made by an observer on the rotating disk, Einstein had this observer put measuring rods around its circumference and along its radius. He argued that because special relativity is valid in a small neighborhood around each point of the rotating disk, an inertial observer on the lower disk will observe each rod placed along the circumference of the upper disk to undergo a Lorentz contraction because it lies along the direction of motion. Therefore, it will take more rods to go around the circumference and the rotating observer will measure the circumference of the rotating disk to be larger than it would have been if the disk had not been rotating.[6] In contrast, because lengths perpendicular to the direction of motion remain unchanged, both the rotating and inertial observers will agree upon the length of the radius of the rotating disk. Reasoning in this way Einstein concluded that an observer on the rotating material disk will measure the ratio of its circumference to its diameter to be greater than π (rather than equal to π as would have been the case if the disk had not been not rotating) and, thus, will conclude that the geometry of the rotating frame is non-Euclidean.

Although Einstein maintained this argument was correct from his first use of it in 1916 (Ref. 7) to his last written discussion of it in 1951,[8] others have found the argument problematic. For example, after analyzing the relativistic effects on the material from which the disk is made, Eddington and Lorentz independently concluded that the ratio of the circumference to the diameter is equal to π and that the geometry on a rotating material disk is Euclidean.[9,10] Similarly, we have had students assert that the geometry of the disk is Euclidean because the circumference of the physical disk should contract by the same amount as the rods used to measure it.

The analyses of Eddington, Lorentz, and our students are part of the large body of literature associated with a paradox posed by Ehrenfest in a short paper published in 1909.[11] Ehrenfest considered a rotating material cylinder and noted that, on the one hand, special relativity predicts that each piece of its circumference will contract in the direction of motion while its radius remains unchanged. Hence, according to special relativity, the ratio of the circumference of the rotating material cylinder to its diameter will be less than π. On the other hand, Euclidean geometry says the ratio must equal π. This apparent contradiction is "Ehrenfest's paradox."[12]

Thus, when investigating the geometry of a rotating material disk Ehrenfest concluded the ratio of the circumference to the diameter is less than π, Eddington and Lorentz (and some of our students) concluded that it is equal to π, and Einstein concluded it is greater than π! Much of the controversy centers around what happens to the material disk—for example, whether it can start from rest and spin up to a relativistic speed without flying apart, and how the stresses and strains caused by rotation affect the density of particles in the material from which the disk is made. Surprisingly, a consensus about the geometry on a rotating material disk has not been reached, as evidenced by recent papers[15,16] and a book[17] devoted solely to examining relativistic physics on rotating disks and in rotating frames of reference.

Consequently, if we explore the transition from special to general relativity in class by following the historical evolution of Einstein's thought, we risk having to discuss the various objections and points of view associated with Ehrenfest's paradox and thus distracting students from the main purpose of the discussion—to motivate the use of non-Euclidean geometry in a theory that generalizes special relativity to include non-inertial frames. To avoid this controversy we have developed another example that does not involve any rotating material body, and it is this example that we present in the next section.

Although many textbooks introduce general relativity using the historical approach we have outlined, many do not. The historical approach is commonly used in sophomore and junior level modern physics books (see, for example, Refs. 18 and 19) and in astronomy and cosmology books (see, for example, Ref. 20), where the purpose is to introduce students to the basic concepts of general relativity and then use them to derive simple experimental predictions such as the gravitational redshift and the bending of light in a gravitational field. Textbooks used in courses which develop the full mathematical theory of general relativity either avoid the historical approach entirely (see, for example, Refs. 21–23) or are very careful about the way in which they present it (see, for example, Ref. 24). The reason the historical approach is not generally used in advanced texts is that although the examples of a uniformly accelerating elevator and a rotating material disk were instrumental in the development of Einstein's thought as he was creating the theory of general relativity, the examples are problematic when analyzed with the mathematical formalism of the completed theory. For example, using the accelerating elevator to predict the bending of light in a gravitational field leads to a result that differs by a factor of 2 from that predicted by the field equations of general relativity (and verified by experiment). Similarly, although the *spatial* geometry of a reference frame traveling on a circular path is non-Euclidean, its *spacetime* geometry is Euclidean. More generally, the accelerating elevator, the rotating material disk, and the example we present in Sec. II all involve *space* and the *local* relationship between space and geometry rather than the fundamental concepts on which the mathematical formalism of general relativity is based—*spacetime* and the *global* relationship between spacetime and geometry. Consequently, whenever the historical approach is used, it should be emphasized that the examples are heuristic, and that although they played an important role in guiding Einstein to the completed theory of general relativity, they are potentially misleading when analyzed with the completed mathematical formalism of that theory.

II. THE APPEARANCE OF NON-EUCLIDEAN GEOMETRY IN NON-INERTIAL FRAMES

Consider a polygon drawn in an inertial frame with N sides of equal length L/N. Suppose a reference frame is moving around the N-gon with a constant nonrelativistic speed u in the counter-clockwise direction as shown in Fig. 1. When an observer at the origin of the moving frame reaches a vertex of the N-gon, the observer can perform an instantaneous (nonrelativistic) boost that makes the x-axis of their frame point along the next segment of the N-gon while keeping the speed of their frame constant.

Alternatively, we can put a uniform magnetic field at each vertex that is perpendicular to the plane of the path, as shown in Fig. 2. In this case, if the reference frame of the traveler is attached to a charged craft, then the magnetic field can be adjusted to turn the craft so that the x-axis of the traveler points in the direction of the next side while keeping the speed of the craft constant. If N is large, then the charged craft will follow a very slight curve at each vertex and, after completing one round trip, the nonrelativistic traveler will conclude that the distance around the N-gon is essentially $N(L/N)=L$. In the limit as $N\to\infty$ the N-gon becomes a circle whose circumference is L, the ratio of the circumference to the diameter D of this circle is $L/D=\pi$, and the geometry is Euclidean.

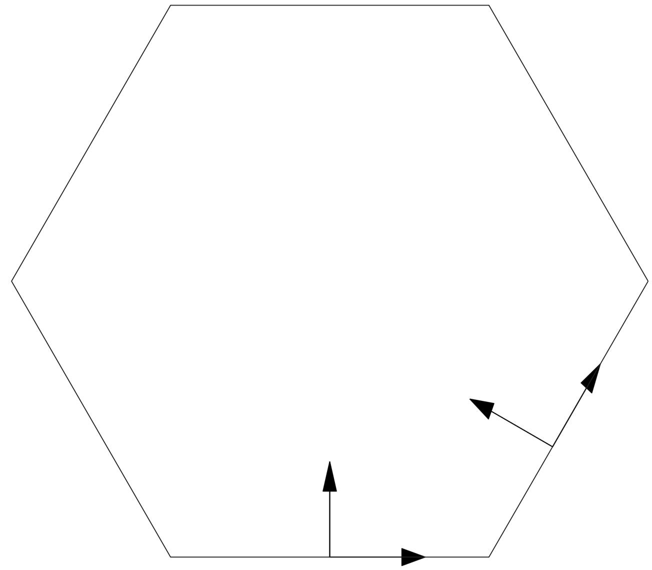

Fig. 1. An N-gon (with $N=6$) drawn in an inertial frame, and a second frame moving around the N-gon in the counter-clockwise direction.

Now suppose a charged craft moves around the N-gon with a relativistic speed v. In this case, because lengths contract in the direction of motion, an observer at the origin of the traveling frame will measure the length of each side of the N-gon to be $L/(N\gamma)$ where $\gamma=(1-v^2/c^2)^{-1/2}$ and c is the speed of light.[25] To understand what happens at each vertex we must consider the effect of acceleration on the γ-factor. As we discussed in Ref. 26, Sec. V, experiments with elementary particles and atomic clocks confirm that the γ-factor is unaffected by acceleration with a constant speed, such as the acceleration of a charged craft moving in a con-

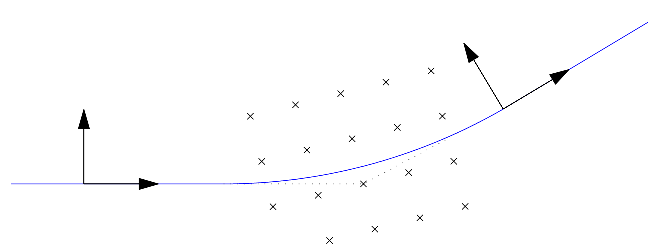

Fig. 2. A reference frame with its x-axis along one side of an N-gon accelerates at a vertex in such a way as to make its x-axis point along the next side by traveling through a constant magnetic field at the vertex which is perpendicular to the plane of the polygon. Although the magnetic field changes the direction of travel of the reference frame it does not change its speed.

stant magnetic field. Thus, if N is large, after one round trip a traveler in the charged craft will conclude that the distance around the N-gon is essentially $N(L/N\gamma)=L/\gamma$ rather than L. In the limit as $N\to\infty$ the N-gon becomes a circle and the relativistic traveler will measure the circumference of the circle to be L/γ, which is less than the circumference L measured by the nonrelativistic traveler. On the other hand, both observers measure the radius of the circle to be the same because lengths perpendicular to the direction of motion are unchanged. Consequently, in the frame of the relativistic traveler, the ratio of the circumference to the diameter is $L/(D\gamma)<L/D=\pi$ and the geometry is non-Euclidean.

When we teach an introductory course this discussion is all that is needed to motivate the appearance of non-Euclidean geometry in non-inertial frames. When we use the historical approach in more advanced courses, we sometimes derive the spatial metric and Gaussian curvature in the traveler's frame. This material is presented in the Appendix.

Our example also can be used to illustrate two interesting properties of clocks at rest in frames traveling with a constant speed on a circular path: any clock in such a frame will run slower than a clock at rest in an inertial frame, and clocks at different locations in such a frame will run at different rates relative to each other.

To prove the first property suppose a clock is located at the origin of the traveling frame and synchronized with a clock in the rest frame of the N-gon when it begins its trip. Because of time dilation, the traveling clock will run slow (as measured by an observer at rest in the frame of the N-gon) as it moves along each side of the N-gon. Hence, if the clock at rest in the frame of the N-gon measures the round trip time of the traveling clock to be T, then the traveling clock will measure the round trip time to be very close to T/γ. (As we have discussed, the γ-factor is not affected by the acceleration with constant speed that occurs at the endpoints of each side.) As $N\to\infty$ and the N-gon becomes a circle, the round trip time measured by the traveling clock becomes exactly T/γ. Experimental confirmations of this prediction are discussed in Ref. 26, Sec. IV.

We would expect the traveling clock to run slower than a clock in the rest frame of the N-gon because both the observer carrying the clock and the observer at rest in the frame of the N-gon must agree upon the relative speed of their frames. The inertial observer measures the relative speed to be $v=L/T$ and, using the previous results, the traveler measures the relative speed to be $(L/\gamma)/(T/\gamma)=L/T$ as expected.[27]

The second property of clocks at rest in a frame traveling on a circular path can be deduced by considering a set of clocks placed at rest along the y-axis of that frame (which points in the radial direction).[28] We have just shown that a clock placed at the origin of a frame moving with a speed v on a circular path of radius R runs more slowly than a clock at rest in the frame of the N-gon by the factor γ. In this case $v=\omega R$ and $\gamma=(1-(\omega R)^2/c^2)^{-1/2}$, where ω is the angular speed of the traveling frame relative to the rest frame of the N-gon. Now consider a set of clocks at rest at different points on the y-axis of the traveling frame. In this case each clock moves in a circle concentric with the circle on which the clock at the origin of the frame is traveling, but with a different radius R. Consequently, even though all of the clocks at rest on the y-axis of the traveling frame have the same angular speed ω with respect to the rest frame of the N-gon, γ is different for each, and thus each clock runs at a different rate relative to the others. An experimental confirmation of this effect is discussed at the beginning of Sec. IV of Ref. 26. As well as being interesting in its own right, this second result can also be used to derive the gravitational redshift.

III. SUMMARY

We have presented an example that shows one way in which generalizing special relativity to a theory that includes non-inertial frames can give rise to a non-Euclidean geometry. The advantage of this example is that it can be used in the same way Einstein used his example of the rotating material disk, but without giving rise to any of the controversy associated with the materiality of that disk.

Because Einstein used the rotating material disk to explore what would be involved in generalizing special relativity to non-inertial frames, he presented it at the beginning of his development of general relativity. Because our example is designed to replace his, it can be presented in the same place in a relativity course.

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APPENDIX: THE METRIC AND GAUSSIAN CURVATURE IN THE TRAVELING FRAME

Consider an N-gon in an inertial frame and another inertial frame moving with a relativistic speed v along one of its sides. If x' and y' are the coordinates of a point on the N-gon as measured in the traveling frame, and x and y are the coordinates of the same point as measured in an inertial frame at rest on the same side of the N-gon, then from the Lorentz transformation we have $x=\gamma(x'+vt')$ and $y=y'$. If two points (x_1',y_1') and (x_2',y_2') are close together, then an observer in the traveling frame will measure $x_2'-x_1'=dx'$ and $y_2'-y_1'=dy'$ when $t_2'=t_1'=dt'=0$. In this case $dx=\gamma dx'$, $dy=dy'$, and the two-dimensional spatial metric in the traveler's frame is

$$ds'^2=dx'^2+dy'^2=(dx/\gamma)^2+dy^2. \tag{A1}$$

When $N\to\infty$ the N-gon becomes a circle of radius R. Therefore, if $\omega=v/R=v/y=v/y'$ is the angular speed[28] of the traveling frame as observed in the frame of the N-gon, then $\gamma=1/\sqrt{(1-w^2y^2/c^2)}$, and

$$ds'^2=\left(1-\frac{\omega^2y^2}{c^2}\right)dx^2+dy^2. \tag{A2}$$

We can describe the meaning of Eq. (A2) as follows: the N-gon is drawn in a two-dimensional manifold in which points are specified by (x,y). An observer at rest with respect to the N-gon measures the distance between two neighboring points as $ds=\sqrt{dx^2+dy^2}$ and concludes that the manifold is Euclidean. In contrast, the traveling observer measures the distance between the same two neighboring points as $ds'=\sqrt{(1-\omega^2y^2/c^2)dx^2+dy^2}$ and concludes that the manifold is non-Euclidean.

The Gaussian curvature K associated with the metric in Eq. (A2) can be calculated using the Theorem Egregium[29] of Gauss as given by Brioschi.[30] The metric in Eq. (A2) has the form

$$ds^2 = E(u,v)du^2 + F(u,v)dudv + G(u,s)dv^2 \tag{A3}$$

with $u=x$, $v=y$, $E(x,y)=(1-\omega^2 y^2/c^2)=1/\gamma^2$, $F(x,y)=0$, and $G(x,y)=1$. For a metric of this form, the Theorem Egregium says the Gaussian curvature is

$$K = -\frac{1}{2\sqrt{EG}}\left[\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{EG}}\frac{\partial G}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{1}{\sqrt{EG}}\frac{\partial E}{\partial v}\right)\right]. \tag{A4}$$

Consequently, in a small neighborhood around any point, the Gaussian curvature of the traveler's space is

$$K = \left(\frac{\gamma\omega}{c}\right)^2\left[1 + \left(\frac{\gamma\omega y}{c}\right)^2\right] = \frac{\omega^2\gamma^4}{c^2}. \tag{A5}$$

Because a Euclidean space has $K=0$, the space of the traveler is non-Euclidean, and because K is greater than zero, the spatial geometry of the traveler's frame is positive and resembles that of a sphere or an ellipsoid. Note that $\omega/c = v/(cR)$ so that in the limit $v/c \to 0$, the Gaussian curvature $K \to 0$ and the spatial geometry of the traveling frame becomes Euclidean as expected.[31]

[a]Electronic mail: msemon@bates.edu
[b]Electronic mail: shimon@sover.net
[c]Electronic mail: swortel@gmail.com

[1]H. A. Lorentz, A. Einstein, H. Minkowski, and H. Weyl, *The Principle of Relativity* (Dover, New York, 1952), pp. 99–108 and 111–118.

[2]J. Stachel, "The rigidly rotating disk as the 'missing link' in the history of general relativity," in *Einstein from B to Z*, edited by J. Stachel (Plenum, New York, 1980), pp. 245–261.

[3]A. Einstein, *Relativity: The Special and the General Theory* (Three Rivers, New York, 1961), pp. 75–95.

[4]A. Einstein, *The Meaning of Relativity* (Princeton U. P., Princeton, NJ, 1956), pp. 55–61.

[5]A. Einstein and L. Infeld, *The Evolution of Physics* (Simon and Schuster, New York, 1966), pp. 209–238.

[6]In this argument Einstein implicitly assumes the material disk is not affected by rotation which, as discussed in the following, many authors have pointed out is incorrect.

[7]Reference 1, pp. 115–117.

[8]Reference 2, pp. 251–252.

[9]E. T. Whittaker, *From Euclid to Eddington* (Cambridge U. P., Cambridge, 1979), pp. 108–111.

[10]A. S. Eddington, *The Mathematical Theory of Relativity* (Cambridge U. P., Cambridge, 1923), pp. 112–113.

[11]See Ref. 17, p. 286.

[12]Apparently it didn't occur to Ehrenfest (nor to anyone but Einstein) that one resolution of the paradox is that the geometry on the disk is non-Euclidean. Interestingly, Pauli (Ref. 13) stated that the conclusion to be drawn from Ehrenfest's argument is that it is impossible to set a material disk into rotation and have it remain rigid according to the definition of rigidity given by Born (Ref. 14).

[13]W. Pauli, *Theory of Relativity* (Pergamon, New York, 1958), pp. 131–132.

[14]See Ref. 2, p. 246.

[15]T. A. Weber, "A note on rotating coordinates in relativity," Am. J. Phys. **65**, 486–487 (1997), and references therein.

[16]See Ref. 17, Chap. 15 for an annotated historical survey.

[17]G. Rizzi and M. L. Ruggier, *Relativity in Rotating Frames* (Kluwer Acameric, Boston, 2004).

[18]J. R. Taylor, C. D. Zafiratos, and M. A. Dubson, *Modern Physics for Scientists and Engineers*, 2nd ed. (Pearson/Prentice Hall, Upper Saddle River, NJ, 2004), pp. 72–79.

[19]S. T. Thornton and A. Rex, *Modern Physics for Scientists and Engineers*, 2nd ed. (Saunders, Philadelphia, 2005), pp. 507–509.

[20]B. Ryden, *Introduction to Cosmology* (Addison-Wesley, Reading, MA, 2002), pp. 27–30.

[21]S. Weinberg, *Gravitation and Cosmology: Applications of the General Theory of Relativity* (Wiley, Hoboken, NJ, 1972).

[22]S. Weinberg, *Cosmology* (Oxford U. P., Oxford, 2008).

[23]R. M. Wald, *General Relativity* (U. of Chicago, Chicago, 1984).

[24]W. Rindler, *Relativity: Special, General and Cosmological*, 2nd ed. (Oxford U. P., Oxford, 2006), pp. 15–27.

[25]The length of each side of the N-gon can be measured by the relativistic traveler using any of the standard methods. For example, two observers in the traveler's frame can measure the location of each end of a side of the N-gon at the same time, or a single observer in the traveler's frame can measure the time necessary for both ends of the N-gon to move by and then multiply this time by the speed at which the side of the N-gon is moving, or an observer in the traveling frame can simply use the Lorentz transformation. Students will have studied these methods earlier in the course as part of the derivation and discussion of length contraction. In the Appendix we use the Lorentz transformation to derive the length of one side of the N-gon as measured in the traveler's frame.

[26]S. Wortel, S. Malin, and M. D. Semon, "Two examples of circular motion for introductory courses in special relativity," Am. J. Phys. **75**, 1123–1133 (2007).

[27]When considering the rotating material disk, Einstein's conclusions about the length of its circumference and the rate of a clock traveling on the circumference don't satisfy this consistency check because he concluded that the rotating observer measures the circumference as γL and the time to complete one rotation as T/γ. The ratio of these two quantities is not equal to the speed $v=L/T$ measured by the inertial observer.

[28]We are only considering points on the y axis of the traveling frame whose speed relative to an inertial observer is $v=\omega R < c$.

[29]Translated variously as "remarkable," "illustrious," "eminent," or "distinguished" theorem. However, as John Rhodes points out, since "ex+grex=out+herd," a more literal translation is "A theorem that sticks out from the herd."

[30]See, for example, ⟨mathworld.wolfram.com/BrioschiFormula.html⟩, ⟨en.wikipedia.org/wiki/Gaussian_curvature⟩, or ⟨en.wikipedia.org/wiki/Theorema_Egregium⟩.

[31]Note that Einstein was mistaken when he concluded that the spatial curvature of a frame traveling on a circular path is hyperbolic, that is, when he concluded that $C/D > \pi$ and $K < 0$.