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Electrodynamics in Terms of Functions Over the Group $SU(2)$: II. Quantization

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In a previous article by two of the present authors Carmeli's group-theoretic method for the formulation of wave equations was applied to the case of the electromagnetic field, and the equations for the vector potential were derived. In the present paper a quantization procedure for these equations is carried out in the Lorentz gauge. It involves two independent variables, corresponding to the number of degrees of freedom of the electromagnetic field in a Hilbert space with a positive-definite metric. Conserved quantities are derived.

1. INTRODUCTION

The present paper is a continuation of a previous one by two of the present authors,(1) which presented a new formalism for the electromagnetic vector potential, representing the equations for the vector potential partially over the space of the group $SU(2)$. The present paper introduces the quantization procedure for these equations and shows its consistency.

Since Carmeli first introduced his group-theoretic approach to the formulation and quantization of wave equations(2) it has proved to be a powerful one in a number of important cases. Carmeli applied his method to Maxwell's equations with and without sources and introduced a gauge-free quantization procedure(3,4) based on quantizing the electric and magnetic fields rather than the vector potential. Carmeli's approach was subsequently applied to the problem of scattering of electromagnetic waves,(5) to the Weyl

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and Dirac equations,[6] and to the linearized equations of general relativity.[7] Recently, a new Hilbert space for quantum gravity was introduced by Carmeli, based on the same method.[8]

The first paper of the present series[1] was motivated by the observation that for the case of interacting matter and electromagnetic fields the quantized theory should be based on the description of the electromagnetic field in terms of the vector potential rather than the electric and magnetic fields. This observation is based on the Aharanov–Bohm effect,[9] most recently discussed by one of the present authors.[10] The first paper[1] introduced a formulation of the equation for the vector potential in the framework of Carmeli's approach. We proceed in the present paper to quantize these equations.

Section 2 includes a review of previous results and correction of some mistakes in Ref. 1. In Section 3 the quantization procedure is carried out. We find that, unlike the case of quantizing the fields using Carmeli's approach,[3] the quantization in terms of the vector potential is not entirely gauge-free. The Lorentz gauge is used. Once this gauge is chosen, however, we have only two components to quantize; no further theoretical structures, such as the Gupta–Bleuler indefinite metric formalism,[11,12] is required. Section 4 includes an analysis of the conserved quantities, which is of interest in itself and shows the consistency of the quantization procedure.

The present quantization in "spherical waves" is appropriate for problems of spherical symmetry, for example, for the extended spherical models of charged particles.[15] Also, in the near field of radiation from single atoms we may sum over spherical waves (this may effect the divergence of perturbation series), for an atom most likely emits spherical-type bursts rather than plane waves!

The usefulness of the present formalism goes beyond application to the case of the electromagnetic field and its interaction with matter. Analogies between the electromagnetic and gravitational fields imply that a progress in quantization methods of the vector potential may indicate a useful approach to quantum gravity. This seems particularly relevant to Carmeli's new Hilbert space for quantum gravity,[8] which is based on the same approach, and was already shown to be applicable to a wide range of Riemannian spacetimes.[13]

2. EXPANDING THE VECTOR POTENTIAL IN THE $SU(2)$ BASIS

In this section we express the vector potential in terms of functions over the group $SU(2)$. We then expand these functions in the basis set provided by the matrix elements of the irreducible representations of $SU(2)$, and derive the equations that determine the expansion coefficients. In this way we review (and in part correct) the work of Ref. 1, and reformulate it in a way that will be useful in the following sections.

Let A_μ represent the electromagnetic vector potential, with A_0 denoting the scalar part and $\mathbf{A}$ the three-vector part. In Heaviside–Lorentz units the Lorentz gauge condition is

$$\partial A_0/\partial t + \nabla \cdot \mathbf{A} = 0 \tag{1}$$

and the vector potentials A_μ satisfy the wave equations

$$\partial^2 A_0/\partial t^2 - \nabla^2 A_0 = \rho \tag{2}$$

$$\partial^2 \mathbf{A}/\partial t^2 - \nabla^2 \mathbf{A} = \mathbf{J} \tag{3}$$

with ρ the charge density, $\mathbf{J}$ the current density, and $\nabla^2\mathbf{A}$ defined by

$$\nabla^2\mathbf{A} \equiv \nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A}) \tag{4}$$

Following Ref. 1, we define

$$\xi_t = A_0 \tag{5}$$

$$\xi_0 = A_r \tag{6}$$

$$\xi_\pm = A_\pm e^{\mp i\phi_2} \tag{7}$$

where

$$A_\pm = -(1/\sqrt{2})(A_\phi \pm iA_\theta) \tag{8}$$

Physically, the components $A_\pm$ are essentially the components of $\mathbf{A}$ along the positive and negative helicity vectors. That is, if one looks along $\hat{\mathbf{r}}$, then $(1/\sqrt{2})(\hat{\mathbf{e}}_\theta \pm i\hat{\mathbf{e}}_\phi)$ are the basis vectors for positive and negative helicities, and $(-iA_+)$ and (iA_-) are the components of $\mathbf{A}$ along these vectors.[2] The variable ϕ_2 was discussed in Ref. 1, as well as the fact that ξ_t, ξ_0, and $\xi_\pm$ are functions over the group $SU(2)$ and so can be expanded in terms of the matrix elements T_{mn} of its irreducible representations. Because of the way in which the ξ's are defined, their expansions reduce to

$$\xi_t = \sum_{j=0}^{\infty} \sum_{m=-j}^{j} \hat{a}^j_{0m}(t, r)\, T^j_{0m}(u) \tag{9}$$

$$\xi_0 = \sum_{j=0}^{\infty} \sum_{m=-j}^{j} a^j_{0m}(t, r)\, T^j_{0m}(u) \tag{10}$$

$$\xi_\pm = \sum_{j=1}^{\infty} \sum_{m=-j}^{j} a^j_{\pm 1,m}(t, r)\, T^j_{\pm 1,m}(u) \tag{11}$$

The expansion coefficients are given by

$$\hat{a}^j_{0m}(t, r) = (2j+1) \int \xi_t(t, r, u)\, T^j_{0m}(u)^*\, du \tag{12}$$

$$a^j_{0m}(t, r) = (2j+1) \int \xi_0(t, r, u)\, T^j_{0m}(u)^*\, du \tag{13}$$

$$a^j_{\pm 1,m}(t, r) = (2j+1) \int \xi_\pm(t, r, u)\, T^j_{\pm 1,m}(u)^*\, du \tag{14}$$

Similarly we can expand the current $j_\mu = (\rho, \mathbf{J})$ as

$$\rho = \sum_{j=0}^{\infty} \sum_{m=-j}^{j} \rho^j_{0m}(t, r)\, T^j_{0m}(u) \tag{15}$$

$$J_r = \sum_{j=0}^{\infty} \sum_{m=-j}^{j} J^j_{0m}(t, r)\, T^j_{0m}(u) \tag{16}$$

and

$$J_\pm = -(1/\sqrt{2})(J_\phi \pm iJ_\theta) = \tilde{J}_\pm e^{\pm i\phi_2}$$

$$\tilde{J}_\pm = \sum_{j=1}^{\infty} \sum_{m=-j}^{j} J^j_{\pm 1,m}(t, r)\, T^j_{\pm 1,m}(u) \tag{17}$$

The expansion coefficients ρ^j_{0m}, J^j_{0m}, and $J^j_{\pm 1,m}$ are determined by equations analogous to (12)–(14).

By inserting the expansions of A_μ and j_μ into the wave equations (2) and (3), we find the differential equations that determine the expansion coefficients of A_μ from those of j_μ. In order to do this, we first express the angular operators in the wave equations in terms of operators acting on functions over $SU(2)$. Following Ref. 1, Eq. (2) becomes

$$\frac{\partial^2}{\partial t^2} \xi_t - \frac{1}{r} \frac{\partial^2}{\partial r^2} (r\xi_t) + \frac{K^2}{r^2} \xi_t = \rho$$

where K^2 is the Casimir operator of $SU(2)$.

Inserting the expansions for ξ_t and ρ, we find that the coefficients $\hat{a}^j_{0m}$ are determined from the equation

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2}\right)(r\hat{a}^j_{0m}) + \frac{j(j+1)}{r^2}(r\hat{a}^j_{0m}) = r\rho^j_{0m} \tag{18}$$

Thus, for a given charge density ρ, the expansion coefficients ρ^j_{0m} can be found, from which Eq. (18) uniquely determines the expansion coefficients for ξ_t.

Next, we find the differential equation that determines a^j_{0m}. The radial part of Eq. (3) is

$$\frac{\partial^2}{\partial t^2}\xi_0 - (\nabla^2\xi_0)_r = J_r \tag{19}$$

Following Ref. 1, we rewrite this equation in terms of operators on functions over $SU(2)$. Using the relation[1]

$$\frac{1}{\sqrt{2}}(K_-\xi_+ - K_+\xi_-) = \frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{\partial A_\phi}{\partial\phi}\right] \tag{20}$$

where K_3, $K_\pm$ are the generators of $SU(2)$, we find that Eq. (19) becomes

$$\frac{\partial^2}{\partial t^2}\xi_0 - \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\xi_0) + \frac{K^2+2}{r^2}\xi_0 + \frac{\sqrt{2}}{r^2}(K_-\xi_+ - K_+\xi_-) = J_r \tag{21}$$

The last term on the left-hand side of Eq. (20) can be expressed in terms of ξ_t by using Eq. (1) (the Lorentz condition), which implies that

$$(K_-\xi_+ - K_+\xi_-) = -\sqrt{2}\left[r\frac{\partial\xi_t}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r^2\xi_0)\right] \tag{22}$$

Combining the derivatives with respect to r, one finds

$$\left(\frac{\partial}{\partial t^2} - \frac{\partial^2}{\partial r^2}\right)(r^2\xi_0) + K^2\xi_0 = r^2 J_r + 2r\frac{\partial}{\partial t}\xi_t \tag{23}$$

Finally, if the expansions for ξ_0, J_r, and ξ_t are inserted into Eq. (23) we obtain

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2}\right)(r^2 a^j_{0m}) + \frac{j(j+1)}{r^2}(r^2 a^j_{0m}) = r^2 J^j_{0m} + 2\frac{\partial}{\partial t}(r\hat{a}^j_{0m}) \tag{24}$$

Thus Eq. (24) shows that the expansion coefficient a^j_{0m} is uniquely determined by the expansion coefficient of the radial component of the current and the time derivative of the expansion coefficient of the scalar potential.

The procedure for deriving the differential equations that determine the angular expansion coefficients $a^j_{\pm 1,m}$ is completely analogous to that for $\hat{a}^j_{0m}$

and a^j_{0m}, except more tedious. As outlined in Ref. 1, the potentials $\xi_\pm$ can be shown to satisfy

$$\frac{\partial^2}{\partial t^2}\xi_\pm - \frac{1}{r^2}\left\{\frac{\partial}{\partial r}\left(r^2\frac{\partial \xi_\pm}{\partial r}\right) - K^2\xi_\pm \mp \sqrt{2}K_\pm \xi_0 \pm \frac{2}{\sin^2\theta}(K_3\xi_\pm \mp \xi_\pm)\right\} = J_\pm \tag{25}$$

Upon substitution of the expansions for $\xi_\pm$, ξ_0, and $J_\pm$ in Eq. (25) one finds

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2}\right)(ra^j_{\pm 1,m}) + \frac{j(j+1)}{r^2}(ra^j_{\pm 1,m}) = rJ^j_{\pm 1,m} + \frac{[2j(j+1)]^{1/2}}{r}a^j_{0m} \tag{26}$$

In summary, in this section we have obtained differential equations that determine the expansion coefficients for components of the electromagnetic vector potential in a basis provided by $SU(2)$. The A_0 equation is

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2}\right)(r\hat{a}^j_{0m}) + \frac{j(j+1)}{r^2}(r\hat{a}^j_{0m}) = r\rho^j_{0m} \tag{27}$$

the A_r equation is

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2}\right)(r^2a^j_{0m}) + \frac{j(j+1)}{r^2}(r^2a^j_{0m}) = r^2J^j_{0m} + 2\frac{\partial}{\partial t}(r\hat{a}^j_{0m}) \tag{28}$$

and the $A_\pm$ equations are

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial r^2}\right)(ra^j_{\pm 1,m}) + \frac{j(j+1)}{r^2}(ra^j_{\pm 1,m}) = rJ^j_{\pm 1,m} + \frac{[2j(j+1)]^{1/2}}{r}a^j_{0m} \tag{29}$$

Once the coefficients are found from Eqs. (27)–(29), the potentials are obtained from Eqs. (9)–(11).

As was noted in Ref. 1, Eqs. (27)–(29) are not independent of each other, but are solved one at a time. First one solves Eq. (27) for $\hat{a}^j_{0m}$, then solves Eq. (28) for a^j_{0m}, and finally uses Eq. (29) to find $a^j_{\pm 1,m}$. Acceptable solutions are subject to the Lorentz gauge condition, which is obtained by substitution of Eqs. (9)–(11) into Eq. (1),

$$\frac{\partial}{\partial t}(r\hat{a}^j_{0m}) + \frac{1}{r}\frac{\partial}{\partial r}(r^2a^j_{0m}) + \left[\frac{j(j+1)}{2}\right]^{1/2}[a^j_{+1,m} - a^j_{-1,m}] = 0 \tag{30}$$

We note that in this approach there are only two independent components of the four potential A_μ. These are found from the real and imaginary parts of $\hat{a}^j_{0m}$. The rest of the expansion coefficients are determined from these, and from the sources **J** through Eqs. (28) and (29). This makes the formalism of $SU(2)$ a particularly convenient one in which to quantize the four-potential A_μ.

3. QUANTIZATION OF THE ELECTROMAGNETIC VECTOR POTENTIAL IN THE $SU(2)$ FORMALISM

In the last section we saw tat in the $SU(2)$ formalism the problem of finding the electromagnetic vector potential A_μ could be reduced to that of solving one partial differential equation, Eq. (27). This equation determines one complex-valued function from which all the usual real components of A_μ can be found. In this section we quantize the free electromagnetic field by quantizing the electromagnetic vector potential as expressed in the $SU(2)$ formalism. The advantage of this approach is that the number of independent variables is equal to the number of independent degrees of freedom, so none of the gauge problems associated with the usual methods of quantizing A_μ arise. Since the equations for A_μ in the $SU(2)$ formalism are similar to those found by Carmeli[3] for the free electromagnetic field $f_{\mu\nu}$, we follow his method of quantization.

We begin by considering Eq. (27) as a complex wave equation, and quantize its field following the canonical procedure. For simplicity define $a^j_m(t, r)$ by

$$a^j_m = r\hat{a}^j_{0m} \tag{31}$$

and denote by $\Box$ the two-dimensional D'Alembertian $\Box = \partial^2/\partial X^\mu \partial X_\mu$, with $\mu = 0, 1$; $X^\mu = (t, r)$; and $X_\mu = (t, -r)$. Using this notation, we can rewrite Eq. (27) as

$$[\Box + j(j+1)/r^2]\, a^j_m = 0, \qquad 0 < r < \infty \tag{32}$$

in the free-field case, with $j = 0, 1, 2,\ldots$ and $m = -j, -j+1,\ldots, j$. An equation similar to (32) can be written for the complex conjugate function a^{j*}_m.

The Lagrangian density for Eq. (32) is

$$L = \sum \omega_j^{-1} L^j_m \tag{33}$$

with

$$L^j_m = \frac{\partial a^{j*}_m}{\partial X^\mu} \frac{\partial a^j_m}{\partial X_\mu} - \frac{j(j+1)}{r^2} a^{j*}_m a^j_m \tag{34}$$

and

$$\omega_j = 2j(j+1)(2j+1)$$

The summation in Eq. (33) is over $j = 0, 1, 2, 3, \ldots$ and $m = -j, -j+1, \ldots, j$. From Eq. (33) we find that the momenta conjugate to a^j_m and a^{j*}_m are

$$\pi^j_m = \partial L/\partial \dot{a}^j_m = \omega_j^{-1} \dot{a}^{j*}_m \tag{35}$$

and

$$\pi^{j*}_m = \partial L/\partial \dot{a}^{j*}_m = \omega_j^{-1} \dot{a}^j_m \tag{36}$$

The Hamiltonian density can now be computed, and we find

$$H = \sum \left[\omega_j \pi^{j*}_m \pi^j_m + \omega_j^{-1} \left(\frac{\partial a^{j*}_m}{\partial r} \frac{\partial a^j_m}{\partial r} + \frac{j(j+1)}{r^2} a^{j*}_m a^j_m \right) \right] \tag{37}$$

If we now assume that a, a^*, π, and π^* are Hermitian operators which satisfy the equal time commutators

$$[a^j_m(t,r), \pi^k_n(t,r')] = [a^j_m(t,r)^*, \pi^k_n(t,r')^*] = i\delta^{jk}\delta_{mn}(r-r') \tag{38}$$

and that all the other commutators vanish, then we have quantized the free electromagnetic field in the $SU(2)$ formalism.

We may next expand our field $a^j_m(r,t)$ as

$$a^j_m(r,t) = \int C^j_m(\omega)\, R^j_m(\omega r)\, e^{i\omega t}\, d\omega \tag{39}$$

where R^j_m satisfy Bessel's equation

$$R^{j''}_m + [\omega^2 - j(j+1)/r^2] R^j_m = 0 \tag{40}$$

so that we choose the solutions regular at the origin. Asymthotically

$$R^j_m(\omega r) \xrightarrow[r\to\infty]{} \frac{1}{\omega r^2} \sin\left(\omega r - j\frac{\pi}{2}\right), \qquad \omega r \gg j \tag{41}$$

$$R^j_m(\omega r) \xrightarrow[\omega r \ll j]{} \frac{1}{r} \frac{(\omega r^j)}{(2j+1)!!} \tag{42}$$

They can be normalized in a spherical box of large radius (R).[14]

It follows then that the creation operators $C^j_m(\omega)$ satisfy

$$[C^j_m(\omega), C^{j'+}_{m'}(\omega')] = \delta_{mn'}\, \delta_{jj'}\, \delta_{\omega\omega'} \tag{43}$$

and create states of definite j, m, and energy ω.

4. CONSERVATION LAWS

Following the canonical procedure, the energy–momentum tensor is defined as

$$T_{\mu\nu} = \sum_{j,m} \frac{\partial L}{\partial(\partial a_m^j/\partial X^\mu)} \frac{\partial a_m^j}{\partial X^\nu} + \frac{\partial L}{\partial(\partial a_m^{j*}/\partial X^\mu)} \frac{a_m^{j*}}{\partial X^\nu} - \eta_{\mu\nu} L \tag{44}$$

with the Lagrangian density L given in Eq. (33) and $\eta_{\mu\nu}$ defined by

$$\{\eta_{\mu\nu}\} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{45}$$

Using the Euler–Lagrange equations

$$\frac{\partial L}{\partial(a_m^j)} - \frac{\partial}{\partial X^\mu} \left(\frac{\partial L}{\partial(\partial a_m^j/\partial X^\mu)} \right) = 0 \tag{46}$$

$$\frac{\partial L}{\partial(a_m^{j*})} - \frac{\partial}{\partial X^\mu} \left(\frac{\partial L}{\partial(\partial a_m^j/\partial X^\mu)} \right) = 0 \tag{47}$$

and the fact that $L = L(a_m^j, a_m^{j*}, \partial a_m^j/\partial X_\mu, \partial a_m^{j*}/\partial X_\mu)$, it is easy to show that

$$\frac{\partial}{\partial X^\nu} T_{\mu\nu} = 0 \tag{49}$$

as expected. Comparison of Eqs. (37) and (44) shows that

$$T_{00} = H \tag{50}$$

is the Hamiltonian density, and

$$T_{01} = \sum_{j,m} \pi_m^j \frac{\partial a_m^j}{\partial r} + \pi_m^{j*} \frac{\partial a_m^{j*}}{\partial r} \tag{51}$$

Again following the canonical procedure, we define the momentum vector

$$P_\nu = \int T_{0\nu}\, dr \tag{52}$$

and note that P_0 is the Hamiltonian and

$$P_1 = \sum_{j,m} \pi_m^j a_m^j + a_m^{j*} \pi_m^{j*} \tag{53}$$

Finally, using the commutation relations (38), it is straightforward to show that

$$i[P_\nu, a^j_m] = \partial a^j_m/\partial X^\nu, \qquad i[P_\nu, a^{j*}_m] = \partial a^{j*}_m/\partial X^\nu \tag{54}$$

Equations (54) show the internal consistency of our quantization procedure for the wave equation (32).

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