Name:____________________________

Mathematics 110 - Fall 2005
Lab Three - Infinity

Goal: to develop a way to decide whether two sets of objects are of the same "size".

Method: We'd like to do this in a way that agrees with our intuitive notion of "number". For example, if we want to know whether a can contains the same number of nuts and bolts, we could simply pair up each nut with a bolt; if every nut is paired with one bolt and every bolt with one nut, then we know there are the same number of each. In the language of mathematics, we would say that the set of bolts and the set of nuts have the same **cardinality**.

Definition: two sets of objects have the same **cardinality** if we can find a one-to-one correspondence between them.

A one-to-one correspondence between two sets requires that

- each object from the first set is paired with exactly one object from the second set
- each object from the second set is paired with exactly one object from the first set

This isn't terribly interesting when applied to sets of finite numbers of objects, but when we begin to look at sets with infinitely many elements, some suprising results appear.

Example. Do the natural numbers and the even natural numbers have the same cardinality?

If they do, then we should be able to find a one-to-one correspondence between these two sets. A diagram will be helpful.

$$\begin{array}{rcccccccccc} \text{Naturals} & 1 & 2 & 3 & 4 & 5 & 6 & \dots & n & \dots \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \\ \text{Even naturals} & 2 & 4 & 6 & 8 & 10 & 12 & \dots & 2n & \dots \end{array}$$

Since we have shown a one-to-one correspondence between these two sets, we can say that they do have the same cardinality (*i.e.*, the same "size").

Is this surprising to you? Do you really believe these sets have the same size even though the evens are the naturals with every second number removed? Just wait - we'll see some even more surprising results in the days ahead.

Exercise 1: Cardinality. Decide whether the following sets have the same cardinality. Explicitly show a one-to-one correspondence or explain why there isn't one.

(a) the even natural numbers and the odd natural numbers

(b) the natural numbers and the natural numbers that are divisible by 3

(c) the natural numbers and the natural numbers with the first 4 elements removed: $\{5, 6, 7, \ldots\}$

(d) the natural numbers and the birth years of everyone alive in the world today

Exercise 2: Generalizing What You've Learned.

(a) Suppose you start with a set A and create a new set B by making a copy of A and removing one object.

(i) Might the set B have the same cardinality as A? Give an example or explain why this isn't possible.

(ii) Might the set B have different cardinality than A? Give an example or explain why this isn't possible.

(a) Suppose you start with a set C and create a new set D by making a copy of C and removing infinitely many objects.

(i) Might the set D have the same cardinality as C? Give an example or explain why this isn't possible.

(ii) Might the set D have different cardinality than C? Give an example or explain why this isn't possible.

Exercise 3: The Hilbert Hotel. This famous hotel is named after the mathematician David Hilbert. It has as many rooms as there are natural numbers, and the rooms are numbered $1, 2, 3, \ldots$

(a) You arrive at the hotel but you haven't reserved a room in advance. The desk clerk says every room is occupied. What can you ask him to do so that you'll get a room and everyone already there will still have one?

(b) You're a very popular person: you have as many friends as there are natural numbers. Late one night, you and all your friends arrive at the Hilbert Hotel when all its rooms are occupied. Can the desk clerk find space for you all if you each insist on a room to yourself and he doesn't want any guest to have to change rooms more than once? Explain why or why not.

Exercise 4: More Cardinality. Decide whether the following sets have the same cardinality. Explicitly show a one-to-one correspondence or explain why there isn't one.

(a) the natural numbers and the perfect squares

(b) the natural numbers and the integers

(c) the even natural numbers and the integers

Exercise 5: Looking Ahead. As we saw in Chapter 2, the set of **rationals** is basically the set of common fractions: all numbers of the form p/q where p and q are integers and q is not 0. Recall that between every pair of consecutive integers (such as 2 and 3) there are infinitely many rationals. Do you think the rationals have the same cardinality as the naturals?