Answers for Homework 3, Math 110.

Chapter 2.3:

23. Lucky 13 Factor. Call the mystery number X. The first statement allows us to express X as 13N + 7, for some whole number N. If we subtract anything less than 7 from this number, we get something not divisible by 13 (something with a remainder of 1, 2, 3, 4, 5 or 6). If we subtract 7, then we get X – 7 = 13N, which is divisible by 13. So the answer is 7.

24. Remainder reminder. As in previous exercise, we write the original number as X = 13N + 7. Adding 22 yields X + 22 = 13N + 29 = 13N + (13 x 2) + 3 = 13(N + 2) + 3. So 13 goes into the new number N + 2 times with a remainder of 3.

25. Remainder roundup. As before, write X = 91N + 72. Then X + 103 = 91N + 155. We recognize that 91 = 7 x 13 and 155 = (22 x 7) + 1, so we can write X + 103 = 7(13N + (7 x 22) + 1) = 13(N + 22) + 1. The final answer is remainder of 1.

26. Related remainders. We write our two numbers, X and Y, in the following way: X = 57N + r, and Y = 57M + r. So (X – Y) = 57N – 57M = 57(N – M) and 57 definitely divides this number. Since 57 = 3 x 19, 3 and 19 will also divide (X–Y). Conjecture: If we divide two numbers by some integer m, the two numbers will have the same remainder upon division if and only if m is a factor of the difference.

Chapter 2.6:

8. Rational or not. \(\sqrt{2}/14\) is the only irrational number in the list. As the ratio of two integers, \(4/9\) is rational by definition, \(1.75 = 7/4\), \(\sqrt{20}/(3\sqrt{5}) = (\sqrt{20}/\sqrt{5})/3 = \sqrt{4}/\sqrt{3} = 2/3\), \(3.14159 = 314,159/100,000\). Also, you could argue that 3.14159 and 1.75 are rational because they each have a terminating decimal expansion.

9. Irrational or not. All are irrational except \(-147 = -147/1\) and \(0 = 0/1\). Notice that \(120/7.5 = 16/10\) and \(\sqrt{16/10} = 4/\sqrt{10}\) which is irrational.

10. \(\sqrt{5}\). The proof is identical to the proof of the irrationality of \(\sqrt{2}\), except the notions even and odd are replaced with divisible by 5 and not divisible by 5, respectively. Assume \(\sqrt{5} = b/c\), with b and c having no common factors. We have \(b^2 = 5c^2\), implying b is divisible by 5 (because 5 is prime and must therefore appear as one of the prime factors of b; we’re using the uniqueness of the prime factorization here). Expressing \(b = 5d\) gives \(25d^2 = 5c^2\), or \(5d^2 = c^2\), implying c is also divisible by 5, a contradiction. This same idea can be applied to the square root of any prime number.

17. An irrational exponent. Assume that \(E\) is rational, that is, \(E = n/m\). Substituting gives \(12^{n/m} = 7\), or \(12^m = 7^n\). Look at the prime factorization of both sides of this equation. Because both sides represent the same number, the prime factorizations must agree (uniqueness of prime factorization). On the left, we have \(n\) 3’s and twice as many 2’s. On the right we have \(m\) 7’s. These two lists can’t be the same, regardless of the values of \(n\) and \(m\). This represents a contradiction, and because our only assumption was the rationality of \(E\), we conclude that \(E\) is irrational.