Math 205A Final Exam (75 points)

Name: Solutions

- Check that you have 9 questions on four pages.
- Show all your work to receive full credit for a problem.
- The last page has some formulas and definitions listed on it.

1. (7 points) Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation with $T\left(\begin{bmatrix} 6 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} 10 \\ 30 \end{bmatrix}$ and $T\left(\begin{bmatrix} 5 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 20 \\ 50 \end{bmatrix}$. Find $T\left(\begin{bmatrix} 9 \\ 8 \end{bmatrix}\right)$. (You may use the following hint: First write $\begin{bmatrix} 9 \\ 8 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 6 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 4 \end{bmatrix}$ and then use properties in the definition of linear transformation.)

$$\begin{bmatrix} 6 & 5 & 9 \\ 5 & 4 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -3 \end{bmatrix}$$

So $\begin{bmatrix} 9 \\ 8 \end{bmatrix} = 4\begin{bmatrix} 6 \\ 5 \end{bmatrix} - 3\begin{bmatrix} 5 \\ 4 \end{bmatrix}$

Hence $T\left(\begin{bmatrix} 9 \\ 8 \end{bmatrix}\right) = 4\,T\left(\begin{bmatrix} 6 \\ 5 \end{bmatrix}\right) - 3\,T\left(\begin{bmatrix} 5 \\ 4 \end{bmatrix}\right)$

$$= 4\begin{bmatrix} 10 \\ 30 \end{bmatrix} - 3\begin{bmatrix} 20 \\ 50 \end{bmatrix}$$

$$= \begin{bmatrix} -20 \\ -30 \end{bmatrix}.$$

2. (9 points) Let $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & -5 \\ -3 & -1 & 1 \end{bmatrix}$.

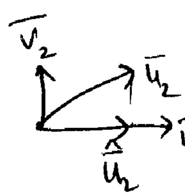
(a) Find a basis for Col A.

$$A \sim \begin{bmatrix} 1 & 0 & .5 \\ 0 & 1 & -2.5 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{Basis for Col } A = \Big\{ \underbrace{\begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}}_{\bar{u}_1}, \underbrace{\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}}_{\bar{u}_2} \Big\}.$$

(b) Find an orthogonal basis for Col A.

$\bar{u}_1 \cdot \bar{u}_2 = 3$. So $\{\bar{u}_1, \bar{u}_2\}$ is not an orthogonal basis

$\bar{v}_1 = \bar{u}_1$



$$\bar{v}_2 = \bar{u}_2 - \hat{u}_2 = \bar{u}_2 - \frac{\bar{u}_2 \cdot \bar{u}_1}{\bar{u}_1 \cdot \bar{u}_1} \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} - \frac{3}{13} \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$$

(simplify further.)

(c) Let $W = ColA$. Find two vectors in $W^{\perp}$.

We need vectors $\bar{u}$ s.t. $\bar{u} \cdot \bar{u}_1 = 0$ and $\bar{u} \cdot \bar{u}_2 = 0$.

Let $\bar{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ $\qquad 2a - 3c = 0$ $\quad a = \frac{3}{2}c$.
$\qquad\qquad\qquad 2b - c = 0$. $\quad$ So $b = \frac{c}{2}$.

(Pick c to be any real number).

So two vectors: $\begin{bmatrix} 3/2 \\ 1/2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$.

3. (9 points) Let $\vec{p}_1(t) = 1 + 4t + t^2$, $\vec{p}_2(t) = 1 - 5t + 4t^2$, $\vec{p}_3(t) = 2 + 8t + 2t^2$ be polynomials in $\mathbb{P}_2$.

(a) Is $\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$ a linearly independent set? Explain.

$\mathbb{P}_2$ ~~isomo~~ isomorphic to $\mathbb{R}^3$. $\mathcal{B}$: standard basis

So $[p_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$, $[\bar{p}_2]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix}$, $[\bar{p}_3]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 8 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 2 \\ 4 & -5 & 8 \\ 1 & 4 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Every column does not have a pivot.

So $\{\bar{p}_1, \bar{p}_2, \bar{p}_3\}$ not lin. ind.

(b) Let $H = \text{Span}\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$. Find a basis $\mathcal{B}$ for H.

From (a), we see that $\{\bar{p}_1, \bar{p}_2\}$ is a lin. ind. set. It also spans H because $\bar{p}_3$ is a multiple of $\bar{p}_1$. So $\mathcal{B} = \{\bar{p}_1, \bar{p}_2\}$.

(c) Let $\vec{p}(t) = 5 + 2t + 11t^2$. Is $\vec{p}$ in H? Explain. If so, write the coordinates of $\vec{p}$ with respect to the basis $\mathcal{B}$ you found in part (b).

$$\begin{bmatrix} 1 & 1 & 5 \\ 4 & -5 & 2 \\ 1 & 4 & 11 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

System is consistent.

So $\bar{p}$ is in H.

$$[\bar{p}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

4. (6 points) $W = \{$all diagonal matrices in $M_{2\times 2}$ with the diagonal entries integers.$\}$. Is W a subspace of $M_{2\times 2}$? Explain.

W is not a subspace of M_{2x2}.

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is in W. Let $c = 0.5$

Then $c\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix}$ is not in W because its diagonal entries cannot be integers.

So W is not a subspace of M_{2x2}.

5. (6 points) Suppose A is a $n \times n$ symmetric matrix.

(a) Show that $(A^2\vec{x}) \cdot \vec{y} = \vec{x} \cdot (A^2\vec{y})$ for any two vectors $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^n$.

$$(A^2\bar{x}) \cdot \bar{y} = (A^2\bar{x})^T\bar{y} = (AA\bar{x})^T\bar{y} = \bar{x}^T A^T A^T \bar{y}$$

$$= \bar{x}^T A A \bar{y} \quad (A^T = A \text{ as } A \text{ is symmetric})$$

$$= \bar{x} \cdot A^2\bar{y}$$

(b) Is A^2 orthogonally diagonalizable? Explain.

$$(A^2)^T = (AA)^T = A^T A^T = AA \quad (\text{since } A^T = A)$$
$$= A^2.$$

So A^2 is symmetric and hence it is orthogonally diagonalizable.

6. (9 points) Suppose U is a 6×6 orthogonal matrix.

(a) Show that $\det U$ is 1 or -1.

$U^{-1} = U^T$. So $UU^T = I$

$\det(UU^T) = 1$

$(\det U)(\det U) = 1$ $(\det(U^T) = \det(U))$

$(\det U)^2 = 1$

$\det U = 1$ or -1.

(b) Is 0 an eigenvalue of U? Explain.

$\det U = 1$ or -1.

So $\det U \neq 0$.

ie $\det(U - 0I) \neq 0$.

So 0 is not an eigenvalue of U.

(c) Do the columns of U form a basis for $\mathbb{R}^6$? Use the two conditions in the definition of a basis to explain your answer.

$\det U \neq 0$.

So U is an invertible matrix.

ie $U \sim I_6$.

Pivot in every column $\Rightarrow$ columns are lin. ind.

Pivot in every row $\Rightarrow$ columns span $\mathbb{R}^6$.

So columns of U form a basis for $\mathbb{R}^6$.

7. (10 points) Define a linear transformation $T : \mathbb{D}_{2\times 2} \to \mathbb{R}^3$ by

$$T\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right) = \begin{bmatrix} a-b \\ a-b \\ 0 \end{bmatrix}.$$

(a) Find $T\left(\begin{bmatrix} -3 & 0 \\ 0 & 0 \end{bmatrix}\right)$.

$$\begin{bmatrix} -3 \\ -3 \\ 0 \end{bmatrix}.$$

(b) Find a matrix that spans the null space of T.

$$T\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a-b \\ a-b \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$a = b.$$

Null space of $T = \text{Span}\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$

(c) Is $\begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$ in the range of T? Explain.

$$T\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a-b \\ a-b \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$$

$$\left.\begin{matrix} a-b=2 \\ a-b=-2 \end{matrix}\right\} \text{inconsistent}$$

So $\begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$ is not in the range.

(d) Is T onto? Explain.

$\begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$ is not in the range of T.

Thus, range $\neq \mathbb{R}^3$, which is the codomain.

Hence T is not onto.

8. (9 points) A 2×2 matrix A has two eigenvalues, 2 and 3. A basis for the eigenspace corresponding to 2 is $\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$ and a basis for the eigenspace corresponding to 3 is $\left\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right\}$.

(a) Is A diagonalizable? Explain.

A is a 2×2 matrix.

$\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ correspond to different evalues.

So they are lin. ind.

Thus A has two lin. ind. evectors.

So A is diagonalizable.

(b) Is $\begin{bmatrix} 5 \\ -5 \end{bmatrix}$ an eigenvector of A? If so, find the corresponding eigenvalue. If not, explain why not.

$\begin{bmatrix} 5 \\ -5 \end{bmatrix} = 5\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. So $\begin{bmatrix} 5 \\ -5 \end{bmatrix}$ is in the eigenspace for 2. So it is an evector and evalue ~~of~~ 2.

OR $A \cdot \begin{bmatrix} 5 \\ -5 \end{bmatrix} = A \cdot 5 \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 5\left(A\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$

$= 5 \cdot 2 \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 2 \cdot \begin{bmatrix} 5 \\ -5 \end{bmatrix}$

So $\begin{bmatrix} 5 \\ -5 \end{bmatrix}$ is an e-vector corresponding to 2.

(c) Is $A - 3I$ an invertible matrix? Explain.

3 is an evalue of A

So $\det(A-3I)=0$.

So $A-3I$ is not invertible.

9. (10 points) The table below gives world hydroelectricity use in tens of thousands of terawatthours. (You do not need more information about this unit of measurement to answer the problem.)

y	1990	1994	2000	2004
h	218	251	271	280

Suppose h and y are approximately related by the equation $b_0 + b_1\, y = h$. Find b_0 and b_1 so that the model $b_0 + b_1\, y = h$ is a least-squares fit to the data. (Start by using the given data to write a system of linear equations to determine b_0 and b_1.)

$$b_0 + (1990)\, b_1 = 218$$
$$b_0 + (1994)\, b_1 = 251$$
$$b_0 + (2000)\, b_1 = 271$$
$$b_0 + (2004)\, b_1 = 280 .$$

$$A = \begin{bmatrix} 1 & 1990 \\ 1 & 1994 \\ 1 & 2000 \\ 1 & 2004 \end{bmatrix} \quad \bar{x} = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} \quad \bar{b} = \begin{bmatrix} 218 \\ 251 \\ 271 \\ 280 \end{bmatrix} .$$

$$A^T A\, \bar{x} = A^T \bar{b} .$$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1990 & 1994 & 2000 & 2004 \end{bmatrix} \begin{bmatrix} 1 & 1996 \\ 1 & 1994 \\ 1 & 2000 \\ 1 & 2004 \end{bmatrix} = \begin{bmatrix} 4 & 7988 \\ 7988 & 15952152 \end{bmatrix}$$

$$A^T \bar{b} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1990 & 1994 & 2000 & 2004 \end{bmatrix} \begin{bmatrix} 218 \\ 251 \\ 271 \\ 280 \end{bmatrix} = \begin{bmatrix} 1020 \\ 2037434 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 7988 & 1020 \\ 7988 & 15952152 & 2037434 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -8249.47 \\ 0 & \cancel{4.26}\ 1 & 4.26 \end{bmatrix}$$

So $b_0 = -8249.47$, $b_1 = 4.26$ } Answers could vary based on round-off error.