Math 205A Final Exam (75 points)

Name: ______________________________

- Check that you have 9 questions on four pages.
- Show all your work to receive full credit for a problem.
- The last page has some formulas and definitions listed on it.

1. (7 points) Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation with $T\left(\begin{bmatrix} 6 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} 10 \\ 30 \end{bmatrix}$ and $T\left(\begin{bmatrix} 5 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 20 \\ 50 \end{bmatrix}$. Find $T\left(\begin{bmatrix} 9 \\ 8 \end{bmatrix}\right)$. (You may use the following hint: First write $\begin{bmatrix} 9 \\ 8 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 6 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 4 \end{bmatrix}$ and then use properties in the definition of linear transformation.)

2. (9 points) Let $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & -5 \\ -3 & -1 & 1 \end{bmatrix}$.

(a) Find a basis for Col A.

(b) Find an orthogonal basis for Col A.

(c) Let $W = Col A$. Find two vectors in $W^{\perp}$.

3. (9 points) Let $\vec{p}_1(t) = 1 + 4t + t^2$, $\vec{p}_2(t) = 1 - 5t + 4t^2$, $\vec{p}_3(t) = 2 + 8t + 2t^2$ be polynomials in $\mathbb{P}_2$.

(a) Is $\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$ a linearly independent set? Explain.

(b) Let $H = \text{Span}\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$. Find a basis $\mathcal{B}$ for H.

(c) Let $\vec{p}(t) = 5 + 2t + 11t^2$. Is $\vec{p}$ in H? Explain. If so, write the coordinates of $\vec{p}$ with respect to the basis $\mathcal{B}$ you found in part (b).

4. (6 points) $W = \{$all diagonal matrices in $\text{M}_{2\times 2}$ with the diagonal entries integers.$\}$. Is W a subspace of $M_{2\times 2}$? Explain.

5. (6 points) Suppose A is a $n \times n$ symmetric matrix.

   (a) Show that $(A^2\vec{x}) \cdot \vec{y} = \vec{x} \cdot (A^2\vec{y})$ for any two vectors $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^n$.

   (b) Is A^2 orthogonally diagonalizable? Explain.

6. (9 points) Suppose U is a 6×6 orthogonal matrix.

   (a) Show that $\det U$ is 1 or -1.

   (b) Is 0 an eigenvalue of U? Explain.

   (c) Do the columns of U form a basis for $\mathbb{R}^6$? Use the two conditions in the definition of a basis to explain your answer.

7. (10 points) Define a linear transformation $T : \mathbb{D}_{2\times 2} \to \mathbb{R}^3$ by

$$T\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right) = \begin{bmatrix} a-b \\ a-b \\ 0 \end{bmatrix}.$$

(a) Find $T\left(\begin{bmatrix} -3 & 0 \\ 0 & 0 \end{bmatrix}\right)$.

(b) Find a matrix that spans the null space of T.

(c) Is $\begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$ in the range of T? Explain.

(d) Is T onto? Explain.

8. (9 points) A 2×2 matrix A has two eigenvalues, 2 and 3. A basis for the eigenspace corresponding to 2 is $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ and a basis for the eigenspace corresponding to 3 is $\left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$.

(a) Is A diagonalizable? Explain.

(b) Is $\begin{bmatrix} 5 \\ -5 \end{bmatrix}$ an eigenvector of A? If so, find the corresponding eigenvalue. If not, explain why not.

(c) Is $A - 3I$ an invertible matrix? Explain.

9. (10 points) The table below gives world hydroelectricity use in tens of thousands of terawatthours. (You do not need more information about this unit of measurement to answer the problem.)

y	1990	1994	2000	2004
h	218	251	271	280

Suppose h and y are approximately related by the equation $b_0 + b_1\, y = h$. Find b_0 and b_1 so that the model $b_0 + b_1\, y = h$ is a least-squares fit to the data. (Start by using the given data to write a system of linear equations to determine b_0 and b_1.)