Name:______________________________

**Mathematics 205**
**Final Exam: C**
**December 16, 2010**

| Problem | Possible | Actual |
|---|---|---|
| 1 | 10 | |
| 2 | 10 | |
| 3 | 20 | |
| 4 | 15 | |
| 5 | 25 | |
| 6 | 20 (5) | |
| Total | 100 | |

You must show all work to receive credit.
No electronic devices other than calculators are permitted.
Give exact answers (such as $\ln 5$ or $e^2$) unless requested otherwise.

1. Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

(a) What does it mean for a mapping to be onto? Is $T(\vec{x}) = A\vec{x}$ onto?

(b) What does it mean for a mapping to be one-to-one? Is $T(\vec{x}) = A\vec{x}$ one-to-one?

2. Suppose $B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & 0 & k \\ 2 & 5 & 6 & 0 \\ 0 & 6 & 4 & 2 \end{bmatrix}$.

(a) If $k = 1$, what is $\det(B)$?

(b) What value of $k$ makes $B$ not invertible?

3. This is a question about linear transformations from $\mathbb{R}^m$ to $\mathbb{R}^n$. Let $T(\vec{x})$ be a linear transformation.

   (a) What are the two properties of a linear transformation?

   (b) Suppose that $\vec{u} \neq \vec{v}$. Does it follow that $T(\vec{u}) \neq T(\vec{v})$? Justify your answer.

   (c) Is it necessarily the case that $T(\vec{0}) = \vec{0}$? Justify your answer.

   (d) Suppose that $\vec{u}$ and $\vec{v}$ are orthogonal. Does it follow that $T(\vec{u})$ and $T(\vec{v})$ are orthogonal? Justify your answer.

4. According to Kepler's first law, a comet should have an elliptic, parabolic, or hyperbolic orbit. In suitable polar coordinates, the position $(r, \theta)$ of a comet satisfies an equation of the form $r = \beta + e(r\dot{c}os\theta)$ where $\beta$ is a constant and $e$ is the eccentricity of the orbit. Suppose the following 5 places are observed:

$$(3, .88), (2.3, 1.1), (1.65, 1.42), (1.25, 1.77), (1.01, 2.14).$$

Using a least-squares approximation, determine the type of orbit. If $e < 1$ the orbit will be an ellipse. If $e = 1$ the orbit will be parabolic. If $e > 1$ the orbit will be hyperbolic.

5. The *trace* of an $n \times n$ matrix $A$ (denoted tr$A$) is the sum of its diagonal entries: $\text{tr}A = \sum_{i=1}^{n} a_{i,i}$. An equivalent way of defining the trace of a matrix is as a sum of its eigenvalues:

$$\text{tr}A = \sum_{i=1}^{n} \lambda_i$$

and note that the $\lambda_i$'s are not necessarily distinct. For example, if A is a $3 \times 3$ matrix that has 2 as an eigenvalue of multiplicity 3, then $\text{tr}A = 2 + 2 + 2 = 6$.

(a) Show that if $A$ and $B$ are similar matrices, then $\text{tr}A = \text{tr}B$.

(b) We may define an inner-product on $\mathcal{M}_{n\times n}$ by $\langle A, B\rangle = \text{tr}A^T B$. Let $A$ be a symmetric matrix and show that $\langle A, A\rangle \geq 0$ and $\langle A, A\rangle = 0$ iff $A = 0$. (Hint: if $A\vec{x} = \lambda\vec{x}$, then $A^2\vec{x} = \ldots$.)

(c) Consider $\mathcal{S}$ and $\mathcal{K}$, the set of symmetric matrices (matrices have the property $A = A^T$) and the set of skew-symmetric matrices (matrices have the property $A = -A^T$). Check if $A \in \mathcal{S}$ and $B \in \mathcal{K}$ then $\langle A, B\rangle = 0$. You may use the facts that $\text{tr}AB = \text{tr}BA$ and $\text{tr}A = \text{tr}A^T$.

(d) Using dim $\mathcal{S} = n(n+1)/2$, dim $\mathcal{K} = n(n-1)/2$, and part (c) explain why we know $\mathcal{M}_{n\times n} = \text{span}\{\mathcal{S}, \mathcal{K}\}$

(e) Write $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ as the sum of a symmetric matrix and a skew-symmetric matrix.

6. Let $\mathcal{D}$ be the set of differentiable functions and $\mathcal{F}$ be the set of functions.

(a) Show that the map $T : \mathcal{D} \to \mathcal{F}$ defined by $T(f) = e^{-t} f'(t)$ is a linear transformation.

(b) What property would eigenvectors of this transformation satisfy? (Hint: it is a differential equation).

(c) **Bonus:** Find the eigenvectors and eigenvalues.