1. An object moving around an ellipse has position at time $t \in [0, 2\pi]$ given $f(t) = (6\sin(t), -4\cos(t))$.

(1A.) In terms of $x$ and $y$, what is the equation of that ellipse parameterized by $f$?

At $t = 0, \frac{\pi}{3}, \pi, \frac{4\pi}{3}$, the object is 
\[(0, 0), (6, 0), (0, -4), (-6, 0),\] respectively. 
Verify this by showing all points fit the form $(6\sin(t), -4\cos(t))$.

The ellipse is $\frac{x^2}{36} + \frac{y^2}{16} = 1$; you can check this by showing all points fit the form $(6\sin(t), -4\cos(t))$.

(1B.) In vector form, what is the equation of the line tangent to this ellipse at $t = \pi/4$?

The point on the ellipse is $f(\pi/4) = (6\cdot\sqrt{2}, -4\cdot\sqrt{2})$.

The velocity vector in general is $f'(t) = (6\cos(t), 4\sin(t))$; at $t = \pi/4$ this is $(6\cdot\frac{\sqrt{2}}{2}, 4\cdot\frac{\sqrt{2}}{2})$.

The velocity vector is in the direction of the line.

So $\vec{L}(t) = (6\cdot\frac{\sqrt{2}}{2}, -4\cdot\frac{\sqrt{2}}{2}) + s(6\cdot\frac{\sqrt{2}}{2}, 4\cdot\frac{\sqrt{2}}{2})$.

(1C.) Find and solve the equation that gives all $t \in [0, 2\pi]$ for which the velocity vector is parallel to the vector $v = (1, 1)$.

So we need the velocity vector to have the form $K(1, 1)$ (we did not say, "when $f'(t) = (1, 1)$")

So we set $f'(t) = K(1, 1)$

$(6\cos(t), 4\sin(t)) = (K, K)$

Then $6\cos(t) = 4\sin(t) \iff \frac{6}{4} = \frac{\sin(t)}{\cos(t)} = \tan(t)$

Also $s = \tan(t)$.

Now, and the $y$-value is $9.828$

So $t = 0.9828 + j\pi$ (since tan is periodic)

For all $j \in \mathbb{Z}$, $t \in [0, 2\pi]$

When $j = 0, 1$, simplify this; get $t = 0.9828$ and $0.9828 + \pi 

\approx 4.1249$.

(1D.) Find and simplify the function that gives the speed of the object at time $t$. (Recall, this is just the length of the velocity vector).

Since $\text{Speed}(t) = ||\text{Velocity}(t)||$, we have

$\text{Speed}(t) = \sqrt{(6\cos(t))^2 + (4\sin(t))^2} = \sqrt{36\cos^2(t) + 16\sin^2(t)}$

Then as a number of ways to write this, we get

$\text{Speed}(t) = 2\sqrt{9\cos^2(t) + 4}$

(1E.) Use (1D) and your math 105 skills to find all $t \in [0, 2\pi]$ at which the speed it maximized. Then identify the positions on the ellipse where this maximum speed is obtained.

The idea was to maximize the speed $\hat{v}$ using $\frac{dv}{dt}$ at critical points:

$(\text{Speed}(t))' = \frac{1}{2} \left( 36\cos^2(t) + 16\sin^2(t) \right)^{1/2} \cdot \left( 72\cos(t)\cdots \frac{1}{2} \right) = 0$.

Now, $\text{Speed}(t)' = 0 \iff t = 0, \pi, 2\pi$ (where $t$ makes the value 0)

Graph of speed for these $t = 0, \pi, 2\pi$ as maxes; corresponds locations on the ellipse

at $(0, -4), (0, 4), \ell (0, -4)$.
2. Suppose that all second partial derivatives of some function \( f : \mathbb{R}^2 \to \mathbb{R} \) exist and are continuous.

2A) In terms of \( f_x \) and \( f_y \), what does it mean to say that \((a, b)\) is a critical point of \( f \)?

It means that \( \left. \begin{array}{l} f_x(a, b) \\ f_y(a, b) \end{array} \right| = 0 \).

2B) In terms of \( f_{xx} \), \( f_{yy} \), and \( f_{xy} \), how does the second derivative test tell us if a critical point \((a, b)\) is a local maximum, local minimum, or saddle point, and under what conditions does the test "fail"?

Note the results here in for \( f : \mathbb{R}^2 \to \mathbb{R} \) with \( D = \left( f_{xx} f_{yy} - (f_{xy})^2 \right) \).

<table>
<thead>
<tr>
<th>( f_{xx} )</th>
<th>( f_{yy} )</th>
<th>( D )</th>
<th>result:</th>
</tr>
</thead>
<tbody>
<tr>
<td>+</td>
<td>+</td>
<td>+</td>
<td>local min</td>
</tr>
<tr>
<td>+</td>
<td>-</td>
<td>=</td>
<td>local max</td>
</tr>
<tr>
<td>-</td>
<td>-</td>
<td>=</td>
<td>saddle</td>
</tr>
</tbody>
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The results are given in this table:

2C) The graph of \( f(x, y) = \sin(x^2) + \sin(y^2) \) accompanies this exam. For this function, \( f_x = 3x^2 \cos(x^2) \), \( f_y = 2y \cos(y^2) \), \( f_{xx} = -9x^4 \sin(x^2) + 6x \cos(x^2) \), \( f_{xy} = -4y^2 \sin(y^2) + 2 \cos(y^2) \), and \( f_{yy} = 0 \).

Find and label on the graph the three points \( a = \left( \frac{3}{2}, \frac{\pi}{2} \right) \), \( b = \left( 0, \frac{\pi}{2} \right) \), and \( c = \left( -\frac{3}{2}, \frac{\pi}{2} \right) \).

2D) Verify that each is a critical point of \( f \).

\[ f(a) = \sin \left( \frac{3}{2} \right) + \sin \left( \frac{\pi}{2} \right) = 1 + 1 = 2 \]

\[ f(b) = \sin 0 + \sin \frac{\pi}{2} = 0 + 1 = 1 \]

\[ f(c) = \sin \left( -\frac{3}{2} \right) + \sin \frac{\pi}{2} = -1 + 1 = 0 \]

2F) Find the value \( k \) of the level curve for each of the three points.

2G) Find the directional derivative in the direction of \( \mathbf{v} = (2, 1) \) at the point \((-0.5, 1)\).

2H) At the point \((-0.5, 1)\), what unit vector points in the direction of maximum increase in \( f \)?

Of points in that direction, so we just need a unit vector in the direction of \( \nabla f \) at \((-0.5, 1)\).

\[ \left\| \nabla f \right\| = \sqrt{\left( 0.74915... \right)^2 + \left( 1.0806... \right)^2} \approx 1.21 \approx \frac{1}{1.21} \left( 0.74915... \right) \approx (0.567..., 0.824...) \]
3A. Let $U$ be an open connected set in $\mathbb{R}^2$. Let $\mathbf{F} : U \rightarrow \mathbb{R}^2$ be a smooth vector field (i.e., the four required partial derivatives of $\mathbf{F}$ are continuous). Let $R$ be a simply connected region in $U$. What equality does Green's theorem give about certain integrals involving $\mathbf{F}$ and $R$? Define any additional terms you use in this answer.

$$\oint_{\partial R} \mathbf{F} \cdot d\mathbf{x} = \iint_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \, dA$$

where $\partial R$ is the boundary of $R$ parametrized in the counter-clockwise direction.

3B. Let $R$ be the region in the first quadrant which is above the line $y = 2$, to the right of the line $x = 1$, and below the parabola $y = 11 - x^2$; you may want to sketch the region. Let $\mathbf{F}(x, y) = ((x - 1)y, (y - 2)x)$. Illustrate Green's theorem for this function $\mathbf{F}$ on $R$, that is, show the two integrals the theorem claims are equal, are in fact equal. Show all your work, and make clear what you are doing, especially when it comes to any necessary parameterizations.

Well do the right-hand side of (3A) first.

$$\iint_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \, dA = \int_{x=1}^{x=3} \left( \int_{y=2}^{y=11-x^2} \left( (y-2) - (x-1) \right) \, dy \right) \, dx$$

$$\int_{x=1}^{x=3} \left( \int_{y=2}^{y=11-x^2} \left( \frac{1}{2} \left( (11-x^2)^{1/2} + x^2 \right) - x(11-x^2) - (11-x^2)^{1/2} \right) \, dy \right) \, dx$$

$$= \int_{x=1}^{x=3} \frac{1}{2} \left( (121 - 22x^2 + x^4) - x(11-x^2) - (11-x^2)^{1/2} \right) \, dx$$

$$= \int_{x=1}^{x=3} \frac{1}{2} x^4 + x^2 - 10x^2 - 9x + \frac{99}{2} \, dx$$

$$= \frac{1}{10} x^5 + \frac{1}{4} x^4 - \frac{10}{3} x^3 - \frac{9}{2} x^2 + \frac{99}{2} x \Big|_1^3$$

$$= \frac{251}{2} - \frac{2521}{60} = \frac{308}{15} = 20.533...$$

Now we need to parameterize $\partial R$ counter-clockwise. We'll require 3 separate integrals:

along $C_1$ ($y=2$) let $\mathbf{r}(t) = (t, 2)$ for $t \in [1, 3]$.

Then $\int_{C_1} F \cdot d\mathbf{x} = \int_{t=1}^{t=3} F(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt = \int_{t=1}^{t=3} (\mathbf{F}(t, 2)) \cdot (1, 0) \, dt$

$$= \int_{t=1}^{t=3} (\mathbf{F}(t, 2)) \cdot (1, 0) \, dt = (\mathbf{F}(t, 2)) \cdot (1, 0) \, |_{t=1}^{t=3}$$

along $C_2$ ($y=11-x^2$) the parametrization $\mathbf{r}(t) = (t, 11-t^2)$ for $t \in [1, 3]$. "Runs backwards", so we use $\mathbf{r}(t) = (t, 11-t^2)$ for $t \in [3, 1]$ instead.

along $C_3$ ($x=1$) the parametrization $\mathbf{r}(t) = (1, t)$ for $t \in [-10, -2]$. Goes the right direction (down)

Then $\int_{C_3} F \cdot d\mathbf{x} = \int_{t=-10}^{t=-2} F \cdot d\mathbf{x} = \int_{t=-10}^{t=-2} (-1, 2t)^{(1, 0)} \, dt = \int_{t=-10}^{t=-2} (-1, 2t) \, dt$

$$= \left[ -t + 2t \right]_{-10}^{-2} = -2 - 30 = -32$$

At last, $\oint_{\partial R} F \cdot d\mathbf{x} = \int_{C_1} F \cdot d\mathbf{x} + \int_{C_2} F \cdot d\mathbf{x} + \int_{C_3} F \cdot d\mathbf{x} = 4 + 18.53 - 32 = 20.53 = \iint_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \, dA$ as expected!
4. Let \( \mathbf{F}(x, y) = (\cos(x)y^2, 2\sin(x)y + 3y^2) \).

4A. Is \( \mathbf{F} \) a path independent vector field? Explain.

\[ \text{It will be, if } \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}. \]

Now, \[ \frac{\partial F_1}{\partial x} = 2\cos(x)y + 0 = 2\cos(x)y \]

and \[ \frac{\partial F_1}{\partial y} = \cos(x)2y = 2\cos(x)y \]

\( \Rightarrow \) since these ARE equal,

yes, \( \mathbf{F}(x, y) \) is path independent.

4B. Find all possible functions \( f \) satisfying \( \nabla f = \mathbf{F} \), or explain why there are none.

Since \( \mathbf{F} \) is path independent, we know \( \mathbf{F} \) is in fact \( \nabla f \) for some scalar valued function \( f \).

Thus, \[ \frac{\partial f}{\partial x} = F_1(x, y) = \cos(x)y^2 \]

so \[ f(x, y) = (\sin(x)y^2 + \phi(y)) \]

Next, \[ \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}((\sin(x)y^2 + \phi(y)) = 2y(\sin(x)) + \phi'(y), \]

and this must equal \( F_2(x, y) \)

so \[ 2y(\sin(x)) + \phi'(y) = 2\sin(x)y + 3y^2 \]

therefore \[ \phi'(y) = 3y^2 \]

so \[ \phi(y) = y^3 + C \] where \( C \) is a constant.

Finally, then, \[ f(x, y) = \sin(x)y^2 + y^3 + C \]

4C. Find \( \iint_R \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA \), where \( R \) is the region in 3B. Make as little work out of it as possible.

By Green's Theorem, this integral equals \( \oint_C \mathbf{F} \cdot d\mathbf{r} \)

But \( \mathbf{F} \) is path independent,

and \( C \) is a closed path.

So this integral is \( 0 \).
5. Consider the circle $C$ of radius 2 on the $xy$ plane centered at the origin. Let $R$ be the region enclosed by this circle. Suppose a vertical wall is constructed all along this circle, and the height $z$ of the wall at any point $(x, y)$ on the circle is $z = 8 - x^2 + y^2$. Let $S$ be the solid object over this circle and inside these walls and having as its "roof" the surface $M$ consisting of the points on the graph of $z = 8 - x^2 + y^2$ which lie over the region $R$. Let $\mathbf{F}(x, y, z) = (x + y, y + z, xy)$. A graph of the surface $M$ over the region $R$ is on the last page of this exam.

Find all of the following. For any which involve integrals, write the standard notation for the integral, eg, $\int_C \mathbf{F} \cdot d\mathbf{x}$ might be one answer. Then set up and simplify to the extent possible (in particular, remove any traces of vectors!) the actual integrals involved, in terms of $s, t$, or whatever variables you use. You do not actually have to evaluate any of the integrals.

5A) a parameterization of the circle $C$, counterclockwise.

As a subset of $\mathbb{R}^3$, we have $\mathbf{g}(t) = (2\cos t, 2\sin t, 0)$

for $t \in [0, 2\pi]$

5B) The integral representing the amount of work done by $\mathbf{F}$ on an object traveling once around this circle, where $\mathbf{F}$ represents a force field.

$$\int_C \mathbf{F} \cdot d\mathbf{x} = \int_0^{2\pi} \mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t) \, dt$$

$$= \int_0^{2\pi} (2\cos t + 2\sin t, 2\sin t + 0, 4\cos t \sin t) \cdot (-2\sin t, 2\cos t, 0) \, dt$$

$$= \int_0^{2\pi} -4\cos t \sin t - 4\sin^2 t + 4 \sin t \cos t + 0 \, dt$$

$$= \int_0^{2\pi} -4\sin^2 t \, dt$$

$$= -4[\pi]$$
5C) A parameterization over some region $R_2$ of the surface $M$, with the normal vectors pointing out of the solid (not inward). Clearly state what your $R_2$ is.

One way:

Let $R_2$ be the disk of radius 2, centered at $(0,0)$.

Top half: $t = \sqrt{4-s^2}$

Bottom: $t = -\sqrt{4-s^2}$

So $s \in [-2,2]$ and given such an $s$, $t \in [-\sqrt{4-s^2}, \sqrt{4-s^2}]$

Let $\vec{g}(s,t) = (s,t,8-s^2+t^2)$

Another way: Let $(s,t) \in [0,2\pi] \times [0,2]$

and consider

let $h(s,t) = (t \cos s, t \sin s, 8-t^2 \cos s) + (t \sin s)^2$.

These "cover" the disk: ""f" gives circles of radius 0 to 2 while $s$ actually goes around these circles.

5D) If $\mathbf{F}$ represents the velocity at the point $(x, y, z)$ of a gas moving through this solid, the flux integral of $\mathbf{F}$ over $M$.

Using the 1st way above,

$$\iiint_{R_2} \mathbf{F} \cdot \hat{n} \, d\sigma = \int_{s=-2}^{s=2} \int_{t=-\sqrt{4-s^2}}^{t=\sqrt{4-s^2}} \mathbf{F}(\vec{g}(s,t)) \cdot \left( \frac{\partial \vec{g}}{\partial s} \times \frac{\partial \vec{g}}{\partial t} \right) \, dt \, ds$$

$$= \int_{s=-2}^{s=2} \int_{t=-\sqrt{4-s^2}}^{t=\sqrt{4-s^2}} (s+t, t+8-s^2+t^2, st) \cdot \left( \frac{1,0,-2s}{2s,-2t,1} \right) \, dt \, ds$$

$$= \int_{s=-2}^{s=2} \int_{t=-\sqrt{4-s^2}}^{t=\sqrt{4-s^2}} (s+t)2s + (t+8-s^2+t^2)(2t) + st \, dt \, ds$$

5E) If $\rho(x, y, z) = x + y$ represents the population density of bacteria living on this surface $M$, the integral representing the total population of bacteria on this surface.

(Note: this is an impossible density function! It's negative in places!)

$$\iiint_{R_2} \rho(x,y,z) \left| \frac{\partial \vec{g}}{\partial s} \times \frac{\partial \vec{g}}{\partial t} \right| \, d\sigma = \int_{s=-2}^{s=2} \int_{t=-\sqrt{4-s^2}}^{t=\sqrt{4-s^2}} (s+t)^2 \sqrt{4s^2 + 4t^2 + 1} \, dt \, ds$$

(since this is $|| (2s,-2t,1) ||$)

5F) Use your calculator program to find a numerical estimate of the integral in (5E), using $N = 15$; CIRCLE your answer. Also write here what functions you used for $Y_1$, $Y_2$, etc. and values for $A$, $B$, etc.

Set $\mathbf{F}_1 = (S+T) \sqrt{4s^2 + 4t^2 + 1}$

$\mathbf{F}_2 = \sqrt{4-s^2}$

$\mathbf{F}_3 = -\sqrt{4-s^2}$

$A = -2$

$B = 2$

$N = 15$

$2.28 \times 10^{12} \; (50, 0)$
for problem 2

for #5