Math 205A Final Exam (100 points)

Name: ______________________________

- Check that you have 10 questions on five pages.
- Show all your work to receive full credit for a problem.

1. (18 points) Short answers: (No explanations needed. Simply write your answers. If you do some calculation to get the answer, show the calculation.)

   (a) (6 points) Let A and B be 4×4 matrices, with $\det A = -4$ and $\det B = 7$. Use properties of determinants to compute:

   (i) $\det A^{-1}$

   (ii) $\det (AB)^T$

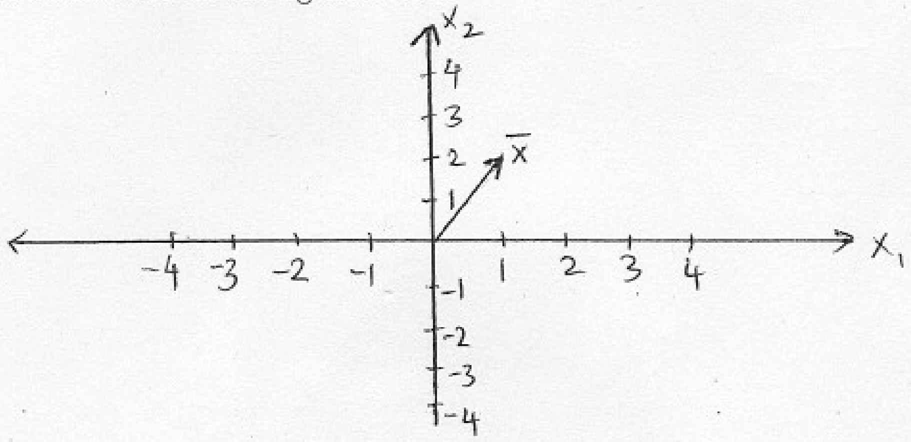
   (b) Suppose C is a 4×3 matrix with a pivot in every column.

   (i) What is the dimension of Nul C?

   (ii) What is the dimension of Col C?

   (c) Define a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(x_1, x_2) = (2x_1 + x_2, x_1 - x_2, x_2)$. Find the standard matrix of T.

(d) The vector $\vec{x}$ shown below is an eigenvector of a 2×2 matrix M corresponding to the eigenvalue -2. Draw $M\vec{x}$ in the figure below.



(e) Let $\vec{x} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$.

(i) Find a unit vector in the direction of the vector $\vec{x}$.

(ii) Find a vector of length 2 in the direction of the vector $\vec{x}$.

2. (9 points) Let $\vec{p}_1(t) = 2 - 3t$, $\vec{p}_2(t) = 1 - t^2$, $\vec{p}_3(t) = 3 - 5t + t^2$.

(a) Verify that these polynomials form a basis for $\mathbb{P}_2$.

(b) Consider the basis $\mathcal{B} = \{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$ for $\mathbb{P}_2$. Find $\vec{q}$ in $\mathbb{P}_2$, given that $[q]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$.

(c) Find the coordinate vector of $\vec{p}(t) = 3 - t + 7t^2$ relative to $\mathcal{B}$.

3. (8 points) A quadratic form on $\mathbb{R}^2$ is given by the formula $Q(x_1, x_2) = 8x_1x_2 - 6x_2^2$.

(a) Find the matrix A of the quadratic form.

(b) Classify the quadratic form.

4. (12 points) Determine if the following sets are subspaces of the appropriate vector spaces. If a set is a subspace, find a basis and the dimension of the subspace. If a set is not a subspace, explain why.

(a) $W = \{\text{all polynomials in } \mathbb{P}_2 \text{ of the form } \vec{p}(t) = at - 3t^2, \text{where } a \text{ is in } \mathbb{R}\}$.

(b) $W = \left\{ \text{all vectors in } \mathbb{R}^3 \text{ of the form } \begin{bmatrix} r+t \\ -s+2t \\ 2r+s \end{bmatrix} \text{ where } r, s, t \text{ are in } \mathbb{R} \right\}$.

(c) $W = \left\{ \text{all matrices in } M_{2\times 2} \text{ of the form } \begin{bmatrix} a & 0 \\ 0 & 2a \end{bmatrix} \text{ where } a \text{ is in } \mathbb{R} \right\}$.

5. (12 points) Let $A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ -1 & 0 & -3 \\ 0 & 1 & -2 \end{bmatrix}$.

(a) Find a basis for Col A. Is the basis you have found an orthogonal set? Explain.

(b) Let $\vec{b} = \begin{bmatrix} 2 \\ 1 \\ -3 \\ 0 \end{bmatrix}$. Find the closest vector to $\vec{b}$ in Col A.

(c) Find a vector in $(\text{Col } A)^{\perp}$.

(d) Find a least-squares solution of $A\vec{x} = \vec{b}$ where $\vec{b}$ is the vector given in part (b).

6. (6 points) Suppose U is an $n \times n$ orthogonal matrix. Let $\vec{x}$ and $\vec{y}$ be any two vectors in $\mathbb{R}^n$. Show that $U\vec{x} \cdot U\vec{y} = \vec{x} \cdot \vec{y}$.

7. (6 points) Let A be a 5×5 matrix with det A not equal to zero. Are the columns of A linearly independent? Explain.

8. (9 points) Define a linear transformation $T : \mathbb{P}_1 \to \mathbb{P}_2$ by $T(a + bt) = bt + (a + b)t^2$.

(a) Find $T(2t)$ and $T(1 - 4t)$.

(b) Find a polynomial p in $\mathbb{P}_1$ such that $T(p) = 3t + 5t^2$.

(c) Is T one-to-one? Explain.

9. (12 points) Let $A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$, $P = \begin{bmatrix} -1 & 4 & -2 \\ 2 & 2 & -1 \\ 0 & 5 & 2 \end{bmatrix}$ and $D = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{bmatrix}$. The matrices P and D diagonalize A, i.e., $A = PDP^{-1}$.

(a) What are the eigenvalues of A? Write an orthogonal basis for the eigenspace corresponding to each eigenvalue of A.

(b) Is P an orthogonal matrix? Explain.

(c) Find an orthogonal matrix Q such that $A = QDQ^T$.

10. (8 points) Consider the following system of equations.

$$\begin{aligned} x_1 + 5x_3 &= 2 \\ -2x_1 + x_2 - 6x_3 &= -1 \\ 2x_2 + 8x_3 &= 6 \end{aligned}$$

(a) Find the general solution of the above system.

(b) Is it possible to change the numbers on the right sides of the equations in the system so that the new system is not consistent? Explain.