Final Exam

Show all your work to receive full credit for a problem. There are a total of 100 points on this test.

1. (3 pts each, 9 points total.) Two trains travel along parallel tracks. The velocity, $v$, of the trains as functions of time $t$ (in hours) are shown in the figure below. Estimate how the following quantities for Train A and Train B compare to each other, and fill in the blank with one of the symbols “$\leq$, $=$, $\geq$”. Briefly explain your reasoning.

(a) Maximum velocity of Train A $\quad$ Maximum velocity of Train B.

(b) Acceleration of Train A at $t = 1$ hour $\quad$ Acceleration of Train B at $t = 1$ hour.

(c) Total distance traveled by Train A $\quad$ Total distance traveled by Train B.
2. (2 pts each, 12 points total) True or False. Tell whether each of the following statements is True or False. Give a one sentence explanation for each of your answers.

(a) If the polynomial function $g$ satisfies $g(0) = -5$ and $g(1) = 2$, then $g$ has a zero somewhere between 0 and 1. ________

(b) If $3 \leq f(x) \leq 5$ for $0 \leq x \leq 2$, then $6 \leq \int_{0}^{2} f(x)dx \leq 10$. ________

(c) $\lim_{x \to \infty} \frac{2x^3 + 5x^2}{3x^3 - 1} = \infty$. ________

(d) The function $h(x) = x^2 + 7$ attains a maximum value on the interval $[-2, 1]$. ________

(e) If a car drives for an hour with an average velocity of 4, and its position can be described by a differentiable function, then there must be a time $t$ between $[0, 1]$ such that the instantaneous velocity at $t$ is exactly 4. ________

(f) $\lim_{x \to 0} \frac{xe^x}{\sin x} = 0$. ________
3. (5 pts each, 20 points total.) Compute the derivatives of the following functions (you don’t need to simplify):

(a) \( f(x) = \ln(\ln(2x^3)) \)

(b) \( g(x) = \arctan(3x^2 + 1) \)

(c) The function \( y \) defined implicitly by \( y + \sin y + x^2 = 9 \). (Solve for \( dy/dx \).)

(d) \( F(x) = \int_0^x \sin(t^2) \, dt \).
4. (16 points total.) Consider $H(x) = \sin x$ on the interval $[0, \pi]$.

(a) (3 pts) Use the axes below to draw a graph of $H(x)$.

(b) (5 pts) Partition the interval into four equal subintervals and use left approximating sums to compute an estimate of $\int_0^\pi \sin x \, dx$.

(c) (5 pts) Evaluate $\int_0^\pi \sin x \, dx$. How does this answer compare to the estimate you found in part (b)?

(d) (3 pts) Explain what you could change in part (b) to get an estimate that is closer to the number you obtained in part (c).
5. (5 pts each, 15 points total.) Evaluate the following integrals.

(a) \[ \int_{1}^{2} (x^2 - x^{-2}) \, dx \]

(b) \[ \int_{1}^{e} \frac{3}{x} \, dx \]

(c) \[ \int_{-2}^{2} x^{101} \, dx \]
6. (3 pts each, 18 points total.) Let \( F(x) = \int_0^x f(t)dt \) where \( f(t) \) is the function shown in the figure. Answer the following questions about \( F \). (You may use estimates if you’re not sure of the exact values.)

(a) What is \( F(1) \)? \( F(3) \)?

(b) What is \( F'(2) \)? What is \( F''(2) \)?

(c) Where are the stationary points of \( F \)?

(d) On what intervals is \( F \) increasing? On what intervals is \( F \) decreasing?

(e) What are the inflection points of \( F \)?

(f) On what intervals is \( F \) concave up? On what intervals is \( F \) concave down?
7. (10 points.) A light is on the ground 40 meters away from a building. Chris, who is 2 meters tall, walks from the light toward the building at 2 meters/second. How rapidly is the size of his shadow on the building changing when he is 20 meters from the building? Is his shadow growing or shrinking at that moment?