Math 106 Section B
Final Exam (100 points)

Name: Solutions

Show all your work to receive full credit for a problem.

There are twelve questions. Questions are printed on both sides of a page.

The following list gives the Taylor series for the functions about $x = 0$:

\[ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \]
\[ \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \]
\[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \]
\[ \ln(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots \]
\[ \frac{1}{1 + x} = 1 - x + x^2 - x^3 + \ldots \]

You can use any of the following facts:
\[ \int \frac{1}{x^p} \, dx \text{ converges for } p > 1 \text{ and diverges for } p \leq 1. \]
\[ \int \frac{1}{x^p} \, dx \text{ converges for } p < 1 \text{ and diverges for } p > 1. \]
\[ \int_0^\infty e^{-ax} \, dx \text{ converges for } a > 0. \]
\[ \sum_{n=1}^\infty \frac{1}{n^p} \text{ converges for } p > 1 \text{ and diverges for } p \leq 1. \]
1. (6 points) Evaluate the following integral exactly (without using the table of integrals):

\[ \int \frac{x}{e^{3x}} \, dx = \int xe^{-3x} \, dx \]

Let \( u = x \) and \( v' = e^{-3x} \), then \( u' = 1 \) and \( v = \frac{-e^{-3x}}{-3} \).

\[ = \frac{x e^{-3x}}{-3} - \int \frac{1}{3} e^{-3x} \, dx \]

\[ = \frac{-xe^{-3x}}{3} + \frac{1}{3} \cdot \frac{e^{-3x}}{-3} + C \]

\[ = \frac{-xe^{-3x}}{3} - \frac{e^{-3x}}{9} + C \]

2. (15 points) Evaluate the following integrals. In case of an improper integral, determine the convergence of the integral. Show clearly any limit computation you do. If the integral converges, find its value.

(a) \( \int \frac{x^2 + 3}{x^2 + 4x + 3} \, dx \)

\[ \int \frac{x^2 + 3}{x^2 + 4x + 3} \, dx = \int \frac{1}{\left(x^2 + 4x + 3\right)} \, dx \]

\[ = \int \frac{1}{\left(x^2 + 4x + 3\right)} \, dx \]

\[ = \int dx + \int \frac{4x}{(x+3)(x+1)} \, dx \]

\[ = x + \frac{1}{-3-(-1)} \left[ -12 \ln |x+3| + 4 \ln |x+1| \right] + C \]

(Using formula 27 in table)

\[ = x + 6 \ln |x+3| - 2 \ln |x+1| + C \]
(b) \[ \int_{-\infty}^{-1} \frac{dt}{2t^2 + 2t + 2} = \int_{-\infty}^{-1} \frac{dt}{(t+1)^2 + 1} = \arctan (t+1)|_{b}^{-1} \]

\[ = 0 - \arctan (b+1) \]

\[ \lim_{b \to -\infty} \int_{b}^{-1} \frac{dt}{t^2 + 2t + 2} = \lim_{b \to -\infty} \left( -\arctan (b+1) \right) \]

\[ = -\left( -\frac{\pi}{2} \right) = \frac{\pi}{2} \]

Integral converges to \( \frac{\pi}{2} \)

(c) \[ \int_{0}^{1/2} \frac{dx}{(2x-1)^2} \] : Improper integral. Denominator is 0 at \( x = \frac{1}{2} \)

\[ \int_{0}^{b} \frac{dx}{(2x-1)^2} = \frac{1}{2} \int_{0}^{b} \frac{dw}{w^2} = \frac{1}{2} \frac{w^{-1}}{-1} + C = -\frac{1}{2} \frac{1}{w} + C \]

\[ w = 2x - 1 \]

\[ dw = 2dx \]

\[ \int_{0}^{b} = -\frac{1}{2} \frac{1}{2x-1} + C \]

\[ b \]

\[ \int_{0}^{1/2} \frac{dx}{(2x-1)^2} = -\frac{1}{2} \left( \frac{1}{2x-1} \right) \bigg|_{0}^{b} = -\frac{1}{2} \left( \frac{1}{2b} + 1 \right) \]

\[ b \to \frac{1}{2}^{-} \]

\[ \int_{0}^{1/2} \frac{dx}{(2x-1)^2} = \lim_{b \to \frac{1}{2}^{-}} -\frac{1}{2} \left( \frac{1}{2b} + 1 \right) \]

\[ = \infty \]

Integral diverges

As \( b \to \frac{1}{2}^{-} \), \( 2b - 1 \to 0^{+} \)

\[ \lim_{b \to \frac{1}{2}^{-}} \frac{1}{2b-1} \]

\[ \frac{1}{2b-1} \to -\infty \]

\[ -\frac{1}{2} \left( \frac{1}{2b-1} + 1 \right) \to +\infty \]
3. (8 points) Find the general solution to the following differential equation:

\[
\cos y \sqrt{9 - x^2} \, dy = x \, dx.
\]

\[
\cos y \, dy = \frac{x}{\sqrt{9-x^2}} \, dx
\]

\[
\int \cos y \, dy = \int \frac{x}{\sqrt{9-x^2}} \, dx
\]

\[
\int \frac{x}{\sqrt{9-x^2}} \, dx = \frac{1}{2} \int \frac{dw}{\sqrt{w}} = -\frac{1}{2} \int \omega^{-1/2} \, d\omega = -\omega^{1/2} + C
\]

\[
= -\sqrt{9-x^2} + C
\]

Thus, \[
\sin y = -\sqrt{9-x^2} + C
\]

\[
y = \arcsin (-\sqrt{9-x^2} + C)
\]

4. (6 points) The following figure shows a bowl of water. If the bowl is full of water, set up an integral that represents the work done to empty the bowl to the 3 ft level. The density of water is 62.4 lb/ft³. (Do not evaluate the integral.)

Volume of slice = \(\pi x^2 \Delta y = \pi (e^y)^2 \Delta y = \pi e^{2y} \Delta y\)

Force on each slice = \(\pi e^{2y} \Delta y \times 62.4\)

Work done on each slice = \(\pi (62.4) e^{2y} \Delta y \cdot (5-y)\)

Total work done = \(\int_3^5 \pi (62.4) e^{2y} (5-y) \, dy\)
5. (9 points) Use the data given in the following table to answer questions about approximations for the integral $\int_0^3 f(x) \, dx$.

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>2</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>9</td>
<td>11</td>
</tr>
</tbody>
</table>

(a) Find $\text{LEFT}(3)$, $\text{LEFT}(6)$ and $\text{TRAP}(6)$.

$\text{LEFT}(3) = 0.5 \left( 2 + 5 + 7 \right) = 14 \quad (3 \text{ subintervals})$

$\text{LEFT}(6) = 0.5 \left( 2 + 4 + 5 + 6 + 7 + 9 \right) = 16.5 \quad (6 \text{ subintervals, each of width 0.5})$

$\text{RIGHT}(6) = 0.5 \left( 4 + 5 + 6 + 7 + 9 + 11 \right) = 21$

$\text{TRAP}(6) = \frac{\text{LEFT}(6) + \text{RIGHT}(6)}{2} = \frac{16.5 + 21}{2} = \frac{37.5}{2} = 18.75$

(b) Can you find $\text{SIMP}(6)$ with the given data? Explain.

$\text{SIMP}(6) = 2 \text{MID}(6) + \text{TRAP}(6)$, $\text{MID}(6)$ cannot be found with the given data as we do not know the values of $f$ at the midpoints of the six subintervals. So $\text{SIMP}(6)$ cannot be found.

(c) Use $\text{LEFT}(3)$ and $\text{LEFT}(6)$ found in part (a) to estimate the actual value of $\int_0^3 f(x) \, dx$.

Number of subintervals increased by a factor of 2.
So error has decreased by a factor of 2.

So $d = 16.5 - 14 = 2.5$

So $\int_0^3 f(x) \, dx = 16.5 + 2.5 = 19$
6. (8 points) Determine the convergence of the following series. Show clearly how you determine the convergence.

(a) \[ \sum_{n=0}^{\infty} \frac{n-n^3}{3n^3 + 1} \]

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} \left( \frac{n-n^3}{3n^3 + 1} \right) = \lim_{n \to \infty} \frac{-n^3}{3n^3} = -\frac{1}{3} \neq 0
\]

So series does not converge.

(b) \[ \sum_{n=1}^{\infty} \frac{n^2-1}{n^2+n^4} \]

\[ \frac{n^2+n^4}{n^2+n^4} > n^4 \]

\[ \frac{1}{n^2+n^4} < \frac{1}{n^4} \]

\[ \frac{n^2-1}{n^2+n^4} < \frac{n^2}{n^4} \]

\[ \frac{n^2-1}{n^2+n^4} < \frac{1}{n^2} \]

\[ \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges.} \]

So by comparison test, \[ \sum_{n=1}^{\infty} \frac{n^2-1}{n^2+n^4} \text{ converges.} \]
7. (8 points) Find the interval and radius of convergence of the series \( \sum_{n=1}^{\infty} \frac{(x-2)^n}{5^n(n+3)} \).

\[
\begin{align*}
\frac{a_n}{a_{n+1}} &= \frac{(x-2)^n}{5^n(n+3)} \quad \text{for} \quad a_n = (x-2)^n \quad \frac{5^n(n+3)}{(n+4)} \quad \text{and} \quad a_{n+1} = (x-2)^{n+1} \quad \frac{5^{n+1}(n+4)}{n+3} \\
\lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| &= \lim_{n \to \infty} \left| \frac{x-2}{5(n+4)} \right| \quad \frac{5^n(n+3)}{(n+4)} \quad \frac{\left| x-2 \right|}{5(n+3)} \\
&= \frac{\left| x-2 \right|}{5}
\end{align*}
\]

By ratio test, series converges if \( \frac{\left| x-2 \right|}{5} < 1 \), i.e., if \( \left| x-2 \right| < 5 \), i.e., if \(-3 < x < 7\).

Series diverges if \( \frac{\left| x-2 \right|}{5} > 1 \)

\[
\begin{align*}
diverges &\quad -3 & converges &\quad 7 & diverges
\end{align*}
\]

Check endpoints:

\( x = -3 \): \( a_n = \frac{(-3-2)^n}{5^n(n+3)} = \frac{(-1)^n}{n+3} \sum_{n=1}^{\infty} \frac{(-1)^n}{n+3} \) converges by alternating series test.

\( x = 7 \): \( a_n = \frac{(7-2)^n}{5^n(n+3)} = \frac{1}{n+3} \sum_{n=1}^{\infty} \frac{1}{n+3} \) diverges (compare to \( \sum_{n=3}^{\infty} \frac{1}{2n} \)).

So interval of convergence: \([ -3, 7 ) \)

radius of convergence = 5.
8. (10 points) Let \( f(x) = x^2 \cos(3x) \). Use this function to answer the following questions:

(a) Write the first four non-zero terms of the Taylor series for \( f(x) \) near \( x = 0 \). Then write out the complete series in summation notation.

\[
\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots
\]

\[
\cos(3x) = 1 - \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} - \frac{(3x)^6}{6!} + \ldots
\]

\[
x^2 \cos(3x) = x^2 \left(1 - \frac{3^2x^2}{2!} + \frac{3^4x^4}{4!} - \frac{3^6x^6}{6!} + \ldots \right)
\]

\[
= \sum_{n \geq 0} (-1)^n \frac{3^{2n} x^{2n+2}}{(2n)!}
\]

(b) Compute \( f^{(200)}(0) \).

\[
f^{(200)}(0) \frac{x^{200}}{(200)!} = (-1)^{99} \frac{3^{200} x^{200 + 2}}{(2 \cdot 198)!}
\]

\[
f^{(200)}(0) \frac{x^{200}}{(200)!} = -\frac{3^{198} x^{200}}{(198)!}
\]

\[
f^{(200)}(0) = -\frac{3^{198} (200)!}{(198)!} = \frac{3^{198}}{2 \cdot 199}
\]

(c) Use the series in part (a) to write the fourth degree Taylor polynomial of \( f(x) \) near \( x = 0 \).

\[
x^2 - \frac{9x^4}{2}
\]
9. (6 points) Find the exact value of the sums of the following series.

(a) $1 + 5 + \frac{25}{2} + \frac{125}{6} + \cdots$

Series for $e^x$ with $x = 5$.

So exact value $= e^5$.

(b) $\frac{7}{4} - \frac{7}{8} + \frac{7}{16} - \frac{7}{32} + \cdots$

$$= 7 \left( \frac{\frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32}}{1 - (-\frac{1}{2})} \right)$$

Geometric series with first term $= \frac{1}{4}$ and common ratio $= -\frac{1}{2}$.

$$= 7 \left( \frac{\frac{1}{4}}{1 - (-\frac{1}{2})} \right)$$

$$= 7 \cdot \frac{1}{4} \cdot \frac{2}{3} = \frac{7}{6}.$$
10. (6 points) The picture below gives the slope field of the differential equation \( \frac{dy}{dx} - y = 2x \).

(a) Plot the solution through the point \((0,1)\).

(b) Use Euler's method with two steps to estimate the value of \(y\) at \(x = 1\) if \(y(0) = 1\) and decide if your answer is too large or too small.

\[
\Delta x = \frac{1-0}{2} = 0.5
\]

<table>
<thead>
<tr>
<th>(x)</th>
<th>(y)</th>
<th>(\Delta y = (\text{slope}) \times \Delta x = (y+2x) \times \Delta x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>((1+0) \times 0.5 = 0.5)</td>
</tr>
<tr>
<td>0.5</td>
<td>1.5</td>
<td>((1.5+1) \times 0.5 = 1.25)</td>
</tr>
<tr>
<td>1</td>
<td>2.75</td>
<td></td>
</tr>
</tbody>
</table>

Too small. (Solution is concave up).
11. (10 points) A colony of wolves in a park has an annual birth rate of 20% and an annual death rate of 15%. Park authorities decide to move 100 wolves every year to another park where they are endangered.

(a) Write a differential equation whose solution is \( P(t) \), the wolf population \( t \) years from now.

\[
\frac{dP}{dt} = 0.2P - 0.15P - 100
\]

i.e. \( \frac{dP}{dt} = 0.05P - 100 \)

(b) Find the general solution of the differential equation.

\[
\frac{dP}{dt} = 0.05P - 100 = 0.05(P - 2000)
\]

\[
\int \frac{dP}{P-2000} = \int 0.05 \, dt \Rightarrow \ln |P-2000| = 0.05t + C
\]

\[
P = 2000 + Ae^{0.05t}
\]

(c) Find and sketch the particular solution if the current wolf population is 3000.

\[
t=0, \quad P=3000 \quad 3000 = 2000 + A \quad \Rightarrow \quad A = 1000
\]

\[
P = 2000 + 1000e^{0.05t}
\]

(d) Find the equilibrium solution and draw it on the sketch in part (c). Is the equilibrium stable or unstable?

\[
P = 2000 \text{ : equilibrium solution.}
\]

Unstable equilibrium.
12. (8 points) Let $P(t)$ denote the population of people infected with a virus after $t$ weeks. Suppose the population obeys the differential equation $\frac{dP}{dt} = \frac{P}{3000}(4000 - P)$ and initially 50 people are infected.

(a) How many people become infected in the long run?

$$\frac{dP}{dt} = \frac{4}{3} P \left(1 - \frac{P}{4000}\right)$$

Compare to $\frac{dP}{dt} = kP(1 - \frac{P}{L})$

People infected in the long run = $L = 4000$.

(b) Sketch the solution $P(t)$.

\[P(t)\]

\[
\begin{array}{c}
4000 \\
2000 \\
50 \\
\end{array}
\]

\[t\]

(c) After about how many weeks does the rate at which people are becoming infected start to decrease?

When $P = 2000$, the rate starts to decrease.

$P = \frac{4000}{1 + Ae^{-\frac{4}{13}t}}$

$t = 0$, $P = 50$

$50 = \frac{4000}{1 + A}$

$A = \frac{4000 - 50}{79} = 79$

$P = \frac{4000}{1 + 79e^{-\frac{4}{13}t}}$

$2000 = \frac{4000}{1 + 79e^{-\frac{4}{13}t}}$

$1 + 79e^{-\frac{4}{13}t} = 2$

\[t = 3.3 \text{ weeks}\]