Math 106 Section B
Final Exam (100 points)

Name: ________________________________

Show all your work to receive full credit for a problem.

There are twelve questions. Questions are printed on both sides of a page.

The following list gives the Taylor series for the functions about $x = 0$:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\ln(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$$

$$\frac{1}{1 + x} = 1 - x + x^2 - x^3 + \cdots$$
1. (6 points) Evaluate the following integral exactly (without using the table of integrals):

\[
\int \frac{x}{e^{3x}} \, dx
\]

2. (15 points) Evaluate the following integrals. In case of an improper integral, determine the convergence of the integral. Show clearly any limit computation you do. If the integral converges, find its value.

(a) \[
\int \frac{x^2 + 3}{x^2 + 4x + 3} \, dx
\]
(b) $\int_{-\infty}^{-1} \frac{dt}{t^2 + 2t + 2}$

(c) $\int_{0}^{1/2} \frac{dx}{(2x - 1)^2}$
3. (8 points) Find the general solution to the following differential equation:

$$\cos y \sqrt{9 - x^2} \, dy = x \, dx .$$

4. (6 points) The following figure shows a bowl of water. If the bowl is full of water, set up an integral that represents the work done to empty the bowl to the 3 ft level. The density of water is 62.4 lb/ft$^3$. (Do not evaluate the integral.)
5. (9 points) Use the data given in the following table to answer questions about approximations for the integral $\int_{0}^{3} f(x) \, dx$.

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>2</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>9</td>
<td>11</td>
</tr>
</tbody>
</table>

(a) Find LEFT(3), LEFT(6) and TRAP(6).

(b) Can you find find SIMP(6) with the given data? Explain.

(c) Use LEFT(3) and LEFT(6) found in part (a) to estimate the actual value of $\int_{0}^{3} f(x) \, dx$. 

6. **(8 points)** Determine the convergence of the following series. Show clearly how you determine the convergence.

(a) \[ \sum_{n=0}^{\infty} \frac{n - n^3}{3n^3 + 1} \]

(b) \[ \sum_{n=1}^{\infty} \frac{n^2 - 1}{n^2 + n^4} \]
7. (8 points) Find the interval and radius of convergence of the series \( \sum_{n=1}^{\infty} \frac{(x - 2)^n}{5^n(n + 3)} \).
8. **(10 points)** Let $f(x) = x^2 \cos(3x)$. Use this function to answer the following questions:

(a) Write the first four non-zero terms of the Taylor series for $f(x)$ near $x = 0$. Then write out the complete series in summation notation.

(b) Compute $f^{(200)}(0)$.

(c) Use the series in part (a) to write the fourth degree Taylor polynomial of $f(x)$ near $x = 0$. 
9. (6 points) Find the exact value of the sums of the following series.

(a) \(1 + 5 + \frac{25}{2} + \frac{125}{6} + \cdots\)

(b) \(\frac{7}{4} - \frac{7}{8} + \frac{7}{16} - \frac{7}{32} + \cdots\)
10. **(6 points)** The picture below gives the slope field of the differential equation \( \frac{dy}{dx} = y - 2x \).

(a) Plot the solution through the point \((0, 1)\).

(b) Use Euler's method with two steps to estimate the value of \( y \) at \( x = 1 \) if \( y(0) = 1 \) and decide if your answer is too large or too small.
11. **(10 points)** A colony of wolves in a park has an annual birth rate of 20% and an annual death rate of 15%. Park authorities decide to move 100 wolves every year to another park where they are endangered.

(a) Write a differential equation whose solution is \( P(t) \), the wolf population \( t \) years from now.

(b) Find the general solution of the differential equation.

(c) Find and sketch the particular solution if the current wolf population is 3000.

(d) Find the equilibrium solution and draw it on the sketch in part (c). Is the equilibrium stable or unstable?
12. (8 points) Let $P(t)$ denote the population of people infected with a virus after $t$ weeks. Suppose the population obeys the differential equation $\frac{dP}{dt} = \frac{P}{3000}(4000 - P)$ and initially 50 people are infected.

(a) How many people become infected in the long run?

(b) Sketch the solution $P(t)$.

(c) After about how many weeks does the rate at which people are becoming infected start to decrease?