Final Exam, Math 205B (Linear Algebra)

This take-home exam is due by 12 noon on **Friday, December 14**. You may consult the textbook (or any other book) and any class notes and handouts, but **please do not discuss any details of this exam with anyone except me!** Please sign the appropriate place on the other side of this sheet and turn it in with your exam. You may ask me questions about the exam, but I reserve the right to give unsatisfying answers. Please show all work.

1. (14 points) Find the complete solution of the system $\begin{pmatrix} 1 & 3 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 1 & 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \\ 3 \end{pmatrix}$.

2. (26 points) Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 10 \\ 3 & 10 & 26 \end{pmatrix}$.

(a) Find the LU factorization of A. Is there anything special about it? Explain. (This might save you some work later.)

(b) Find the determinant of L and the determinant of U.

(c) Use your answer to (a) to solve $A\vec{x} = \begin{pmatrix} 9 \\ 31 \\ 83 \end{pmatrix}$.

(d) Use your answer to (b) to find the determinant of A.

(e) Find L^{-1} and U^{-1}.

(f) Use your answer to (e) to find A^{-1}.

3. (19 points) Find a basis for each of the four subspaces associated with the matrix

$$A = \begin{pmatrix} 1 & 1 & 2 & -2 \\ 1 & 2 & 1 & 5 \\ 2 & 1 & 1 & -3 \\ 4 & 7 & 9 & 5 \end{pmatrix}$$

What is the factored form of A that displays these bases?

4. (14 points) (a) Evaluate the determinant $\begin{vmatrix} 1 & 2 & 1 & 3 \\ 1 & 3 & 1 & 2 \\ 2 & 2 & 1 & 4 \\ 3 & 2 & 3 & 2 \end{vmatrix}$.

(b) Are the vectors $\begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 3 \\ 2 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \\ 4 \\ 2 \end{pmatrix}$ linearly independent? Explain.

5. (14 points) Explain how you can tell that $R = \frac{1}{5}\begin{pmatrix} -1 & 2 & 2 & 4 \\ 2 & 1 & -4 & 2 \\ 2 & -4 & 1 & 2 \\ 4 & 2 & 2 & -1 \end{pmatrix}$ is a reflection matrix. Find a basis for the subspace S of $\mathbb{R}^4$ that R reflects through, and a basis for $S^{\perp}$.

6. (8 points) Suppose P is a projection matrix onto some k-dimensional subspace K of $\mathbb{R}^n$. Describe the eigenvalues and eigenvectors of P as well as you can.

7. (30 points) Let $A = \begin{pmatrix} 5 & 2 \\ 2 & 8 \end{pmatrix}$. (What happens in this problem also happens for a generic symmetric matrix, more or less, but this example is particularly nice.)

(i) Calculate the eigenvalues λ_1 and λ_2 of A, and the corresponding eigenvectors $\vec{v}_1$ and $\vec{v}_2$.

(ii) Calculate the projection matrix P_1 onto $\vec{v}_1$, and the projection matrix P_2 onto $\vec{v}_2$.

(iii) What is $P_1 + P_2$? (The answer should be nice.)

(iv) What is $\lambda_1 P_1 + \lambda_2 P_2$? (The answer should be nice.)

(v) Calculate A^{-1}.

(vi) What is $\frac{1}{\lambda_1} P_1 + \frac{1}{\lambda_2} P_2$? (The answer should be nice.)

(vii) Calculate $P_1 P_2$ and $P_2 P_1$. (The answers should be nice.)

(viii) If n is a positive integer, prove that $A^n = \lambda_1^n P_1 + \lambda_2^n P_2$. Hint: $A^n = (\lambda_1 P_1 + \lambda_2 P_2)^n$, which simplifies a great deal because of (vii).

(ix) Explain why the result of (viii) holds also for $n = 0$.

(x) If n is a positive integer, prove that $A^{-n} = \left(A^{-1}\right)^n = \lambda_1^{-n} P_1 + \lambda_2^{-n} P_2$. Hence the result of (viii) holds for *all* integers n.

(xi) Does the result of (viii) hold also for $n = \frac{1}{2}$? In other words, can you use it to calculate $\sqrt{A}$? Explain. Are there other powers of A that you can calculate this way?

I affirm that I did not receive help from another person in doing this exam, nor did I give help to another student in the class.

(signed) ______________________________