Solutions to Fall 2011 Math206 exam

(1a) $2x^2-3y^2-z=0 \longrightarrow z=2x^2-3y^2$

$\boxed{\text{hyperbolic paraboloid}}$ $\boxed{iv}$

level curves are either W or Y

when $z=1$ $2x^2-3y^2=1$

if $x=0$ then $-3y^2=1$

but this has no real sol'ns

thus the $z=1$ level curve doesn't intersect the y-axis $\boxed{Y}$

(1b) $4x^2+4y^2=z^2$ $\boxed{\text{cone}}$ $\boxed{ii}$

Level curves are circles b/c both x^2 and y^2 have equal coeff.

So either U or V.

$\boxed{U}$ b/c $z=1$ and $z=-1$ are the same level set

(1c) $3x^2+y^2+4z^2=1$ $\boxed{\text{ellipsoid}}$ $\boxed{V}$

Level sets are ellipses

in particular at $z=0$ $3x^2+y=1$ is an ellipse

not a point, so $\boxed{X}$

(2a) let $x=0=y$ then $z=12$ so $\boxed{(0,0,12)}$ is on plane

(2b) read the normal vector from the coefficients $\boxed{(2,-3,1)=\vec{n}}$

(2c) a vector // to the plane connects any 2 points on the plane

$x=0=z \Rightarrow -3y=12,\ y=-4$ so $(0,-4,0)$ is on plane

and $\vec{v}=(0,0,12)-(0,-4,0)=\boxed{(0,4,12)}$ is //.

(2d) This answer varies depending on your answer to (2c).

$$(2,-3,1)\times(0,4,12)=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 2 & -3 & 1\\ 0 & 4 & 12\end{vmatrix}=\begin{vmatrix}-3 & 1\\ 4 & 12\end{vmatrix}\hat{\imath}-\begin{vmatrix}2 & 1\\ 0 & 12\end{vmatrix}\hat{\jmath}+\begin{vmatrix}2 & -3\\ 0 & 4\end{vmatrix}\hat{k}$$

$$=-40\hat{\imath}-24\hat{\jmath}+8\hat{k}$$

(3) $U_s = \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s} = 2 \cdot 4 + 3 \cdot 6 = \boxed{26}$

(4) $D\vec{f}(-1,1) = \begin{bmatrix} 2y & 2x \\ 3 & -2y \end{bmatrix}\Bigg|_{(-1,1)} = \boxed{\begin{bmatrix} 2 & -2 \\ 3 & -2 \end{bmatrix}}$

(5) $\vec{\nabla} f = (3y - 3x^2, 3x - 3y^2) = \vec{0}$

$3y - 3x^2 = 0 \quad , \quad 3x - 3y^2 = 0$

$y = x^2 \qquad x = y^2$

$x = x^4 \rightarrow x^4 - x = 0$

$x(x^3 - 1) = 0 \longrightarrow x = 0, 1$

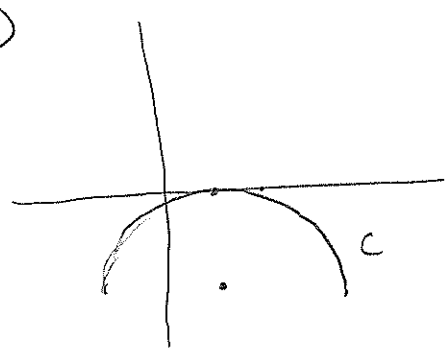
so critical points are $(0,0)$ and $(1,1)$.

$Hf = \begin{bmatrix} -6x & 3 \\ 3 & -6y \end{bmatrix}$

$Hf(0,0) = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}$ $\left.\begin{array}{l} \det[0] = 0 \\ \det\begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} = -9 < 0 \end{array}\right\} \Rightarrow$ $\boxed{\text{saddle point}}$ at $(0,0)$

$Hf(1,1) = \begin{bmatrix} -6 & 3 \\ 3 & -6 \end{bmatrix}$ $\left.\begin{array}{l} \det[-6] = -6 < 0 \\ \det\begin{bmatrix} -6 & 3 \\ 3 & -6 \end{bmatrix} = 27 > 0 \end{array}\right\} \Rightarrow$ $\boxed{\text{local max}}$ at $(1,1)$

(6a)



$\vec{f}(t) = (2\cos t + 1, \ 2\sin t - 2)$

$0 \le t \le \pi$

6b

$$\int_C h\,dL = \int_0^{\pi} h(\vec{f}(t))\,\|\vec{f}\,'(t)\|\,dt$$

$$= \int_0^{\pi} \left[(2\cos t+1)^2 + (2\sin t-2)+20\right] \|(-2\sin t, 2\cos t)\|\,dt$$

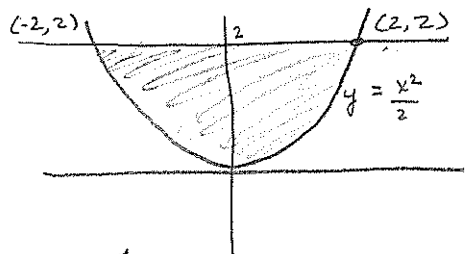
$$= \int_0^{\pi} \left[(2\cos t+1)^2 + (2\sin t-2)+20\right] \sqrt{4\sin^2 t + 4\cos^2 t}\;dt$$

$$= 2\int_0^{\pi} \left[(2\cos t+1)^2 + (2\sin t-2)+20\right] dt$$

7

$x = \sqrt{2y} \rightarrow x^2 = 2y$

$x = -\sqrt{2y} \rightarrow x^2 = 2y$



$$\boxed{\int_{-2}^{2}\int_{x^2/2}^{2} (3x+2y)\,dy\,dx}$$

8a check $\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y} = (9x^2y^2+4y^3) - (9x^2y^2+4y^3) = 0$

so $\vec{F}$ is path independent.

8b First find the potential function f s.t. $\vec{\nabla} f = \vec{F}$

$f_x = 3x^2y^3 + y^4, \qquad f_y = 3x^3y^2 + y^4 + 4xy^3$

$f = \int (3x^2y^3+y^4)\,dx = \int (3x^3y^2+y^4+4xy^3)\,dy$

$f = x^3y^3 + xy^4 + g_1(y) = x^3y^3 + \frac{y^5}{5} + xy^4 + g_2(x)$

$\Rightarrow g_1(y) = \frac{y^5}{5} + C$

so $f = x^3y^3 + xy^4 + \frac{y^5}{5} + C$

(Note: finding a potential function is also enough to show that $\vec{F}$ is path independent.)

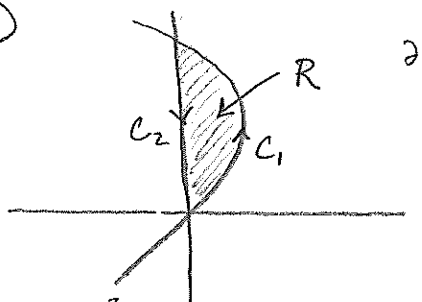
(8b) continued

$$\int_C \vec{F}\cdot d\vec{x} = f(-1,3) - f(1,2)$$

$$= \left((-1)^3(3)^3 + (-1)(3)^4 + \frac{3^5}{5}\right) - \left((1)^3(2)^3 + (1)(2^4) + \frac{2^4}{5}\right)$$

$$= -89.8$$

(9)



$\partial R = C = C_1 \cup C_2$

problem asks us to calculate both $\oint_{\partial R} \vec{F}\cdot d\vec{x}$ and $\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$ and confirm that they are equal.

C_1: $\vec{f}_1(t) = (-t^2+t, t)$; $0 \le t \le 1$

$$\int_{C_1} \vec{F}\cdot d\vec{x} = \int_0^1 (-t^2+t+t, (-t^2+t)t)\cdot(-2t+1, 1)\,dt$$

$$= \int_0^1 (t^3 - 4t^2 + 2t)\,dt = \frac{t^4}{4} - \frac{4}{3}t^3 + t^2\Big]_0^1 = \frac{-1}{12}$$

C_2: $\vec{f}_2(t) = (0, 1-t)$; $0 \le t \le 1$

$$\int_{C_2} \vec{F}\cdot d\vec{x} = \int_0^1 (1-t, 0)\cdot(0,-1)\,dt = \int_0^1 0\,dt = 0$$

so $\oint_{\partial R} \vec{F}\cdot d\vec{x} = -\frac{1}{12} + 0 = \boxed{-\frac{1}{12}}$

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \int_{y=0}^1 \int_{x=0}^{-y^2+y} (y-1)\,dx\,dy = \int_0^1 (yx - x)\Big]_0^{-y^2+y} dy$$

$$= \int_0^1 \left(y(-y^2+y) - (-y^2+y)\right) dy = \int_0^1 (-y^3 + 2y^2 - y)\,dy = -\frac{1}{4}y^4 + \frac{2}{3}y^3 - \frac{1}{2}y^2\Big]_0^1 = \boxed{\frac{-1}{12}}$$