Some of the following may be useful.

$$\begin{bmatrix} 27 & 9 & 1 & 47 \\ 8 & 4 & 1 & 10 \\ -1 & 1 & 1 & -5 \\ -8 & 4 & 1 & -38 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 47 & 27 & 9 & 1 \\ 10 & 8 & 4 & 1 \\ -5 & -1 & 1 & 1 \\ -38 & -8 & 4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 114 & 20 & 55 \\ 20 & 4 & 14 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -15/14 \\ 0 & 1 & 62/7 \end{bmatrix} \qquad \begin{bmatrix} 70 & 14 & 55 \\ 14 & 4 & 20 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5/7 \\ 0 & 1 & 15/2 \end{bmatrix}$$

$$\begin{bmatrix} 14 & 4 & 20 \\ 70 & 14 & 55 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5/7 \\ 0 & 1 & 15/2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 4 & 0 & 2 & 1 & 0 & 0 & -1 \\ 0 & 1 & -3 & 0 & 5 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 3 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 & -11 & -4 & -2 & 12 \end{bmatrix} \sim \begin{bmatrix} 6 & 0 & 24 & 2 & 14 & 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 & 5 & 0 & 1 & 0 & 0 \\ -3 & -2 & -6 & 1 & -15 & 0 & 0 & 1 & 0 \\ 5 & 0 & 20 & 2 & 12 & 0 & 0 & 0 & 1 \end{bmatrix}$$

1. There is a cubic polynomial of the form $Ax^3 + Bx^2 + C$ which contains the four points $(3, 47)$, $(2, 10)$, $(-1, -5)$ and $(-2, -38)$.

1A. Set up the system of equations you need to solve in order to find A, B and C.

1B. Solve the system, and find the polynomial. This is not a least squares problem!

2. The points $(1,8)$, $(2,5)$, $(4,4)$ and $(7,3)$ do not lie on any one line. Find m and b so that $y = mx + b$ is the least squares line that best fits these four points. Show all your work.

3. Suppose W is the column space of $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$ and let $\mathbf{z} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$.

3A. What equation(s) does $\mathbf{z}$ have to satisfy in order for $\mathbf{z}$ to be in $W^{\perp}$?

3B. What matrix equation represents the answer to (3A)?

3C. Use (3B) to find a basis for $W^{\perp}$.

3D. Let S be the set of those two column vectors in A. Note that S is orthogonal. Produce an ortho*normal* set of vectors which also spans W.

3E. Use S and our dot "product formula" to find the projection $\mathbf{w}$ of $\mathbf{y} = \begin{bmatrix} 5 \\ 3 \\ 4 \\ 2 \end{bmatrix}$ onto W.

3F. What is the distance from $\mathbf{w}$ to $\mathbf{y}$?

4. Let $A = \begin{bmatrix} 3 & -8 & 6 \\ 3 & -7 & 3 \\ 3 & -4 & 0 \end{bmatrix}$; then A is diagonalizable. Find P and D with the "right properties".

Facts: One eigenvalue for A is 2 and one eigenvector for A is $\begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$.

Show all your steps!

5. Define $T : \mathbf{P}_4 \to \mathbf{P}_4$ by $T(p(x)) = p'(x)$; then T is a linear transformation (LT).

5A. Find (describe) the kernel of this LT.

5B. Is $1 + x + x^2 + x^3$ in the image of T? Explain.

5C. Is $1 + 5x^4$ in the image of T? Explain.

6. Suppose that $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ is a set of vectors all having length 2, and that W=span(S). Suppose furthermore that $S \cup \{\mathbf{b}\} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p, \mathbf{b}\}$ is orthogonal, and $\mathbf{b}$ is not equal to any of the $\mathbf{u}$'s.

6A. Show that $\mathbf{b}$ is perpendicular to *any* vector in W, and hence is in $W^\perp$.

6B. Is S a basis for W? Explain.

6C. Can $\mathbf{b}$ be a linear combination of the members of S? Explain any possiblilities.

7. Let $G = \begin{bmatrix} 6 & 0 & 24 & 2 & 14 \\ 0 & 1 & -3 & 0 & 5 \\ -3 & -2 & -6 & 1 & -15 \\ 5 & 0 & 20 & 2 & 12 \end{bmatrix}$ and let $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$.

7A. Is every such $\mathbf{b}$ in the column space of G? If not, describe any conditions that must be met.

7B. What is the dimension of the column space of G?

7C. Find all solutions of $G\mathbf{x} = \begin{bmatrix} 28 \\ -15 \\ 2 \\ 21 \end{bmatrix}$. Express your answer in terms of a particular solution and the solutions of the corresponding homogeneous equation.

7D. What is the dimension of the null space of G?

7E. Find a basis for the null space of G.

7F. Find a basis for the row space of G.

8. For each of the following vector spaces, give an *two* obvious bases with no common vectors, or explain why there is no basis. Also then give the *dimension* of the vector space:

8A. $\mathbf{P}_3$

8B. $\{\mathbf{0}\}$

8C. $\mathbf{R}^4$

8D. The space $\mathcal{S}$ of all sequences $(s_1, s_2, s_3, \ldots)$ of real numbers, with addition and scalar multiplication as discussed in class.

9. Suppose $B \in M_{4\times 6}$ and its RREF form has at least one row of zeros.

9A. What are the maximum and minimum possible dimensions for $\text{Nul}(B)$? MAX: MIN:

9B. What are the maximum and minimum possible values for the rank of B? MAX: MIN:

10. Suppose $D = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 8 \\ 2 \end{bmatrix}$; then the least squares solution to $D\mathbf{x} = \mathbf{b}$ is $\begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ where x_3 is free.

10A. Do the columns of D form orthogonal set? Explain.

10B. Find the projection of $\mathbf{b}$ onto the column space of D. Show your work.