YOUR GRADE IS BASED ON THE PROCESS AS WELL AS THE FINAL RESULT. SHOW ALL YOUR STEPS CLEARLY SO YOU WILL BE ELIGIBLE FOR THE MOST PARTIAL CREDIT. YOU MAY USE A CALCULATOR, BUT NO NOTES, BOOKS, OR OTHER STUDENTS. GOOD LUCK!

1.) (10 pts.) For \( f(x) = 2x^2 - 3 \), compute \( f'(x) \) using the limit definition of the derivative.

\[
\begin{align*}
    f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
    &= \lim_{h \to 0} \frac{(2(x+h)^2 - 3) - (2x^2 - 3)}{h} \\
    &= \lim_{h \to 0} \frac{2x^2 + 4xh + 2h^2 - 3 - 2x^2 + 3}{h} \\
    &= \lim_{h \to 0} \frac{4xh + 2h^2}{h} \\
    &= \lim_{h \to 0} \frac{h(4x + 2h)}{h} \\
    &= \lim_{h \to 0} (4x + 2h) \\
    &= 4x
\end{align*}
\]
2. (15 pts.) Two people start biking from the same point. One bikes east at 15 mph, the other south at 18 mph. How fast is the distance between the two people changing after 20 minutes? (Give your answer in exact terms.)

We know:

\[
\frac{dx}{dt} = 15 \text{ mph}
\]

\[
\frac{dy}{dt} = 18 \text{ mph}
\]

After 20 min, \( x = \frac{1}{3}(15) = 5 \text{ mi} \) (1/3 hr)

\( y = \frac{1}{3}(18) = 6 \text{ mi} \)

\[
D = \sqrt{x^2 + y^2} = \sqrt{61} \text{ mi}
\]

Relationship of variables:

\[
x^2 + y^2 = D^2
\]

Derivative w.r.t. time:

\[
2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2D \frac{dD}{dt}
\]

Substitute in numbers:

\[
2(5)(15) + 2(6)(18) = 2(\sqrt{61}) \frac{dD}{dt}
\]

Solve for \( \frac{dD}{dt} \):

\[
\frac{dD}{dt} = \frac{183}{\sqrt{61}} \text{ mph} \quad (\approx 23.43 \text{ mph})
\]

Exact terms
3.) (15 pts.) Given the integral \( \int_{1}^{4} \frac{5}{x} \, dx \),

a.) (4 pts.) compute the exact integral using the Fundamental Theorem of Calculus;

For \( f(x) = \frac{5}{x} \), an antiderivative is \( F(x) = 5 \ln x \).

Then \( 5 \ln 4 - 5 \ln 1 = 5 \ln 4 \)

\( = 0 \)

b.) (4 pts.) approximate the integral using a trapezoidal sum with 3 subintervals (include a picture);

\[ T_3 = \frac{1}{2} \left( \frac{5}{2} \right)(1) + \frac{1}{2} \left( \frac{5}{2} + \frac{5}{3} \right)(1) + \frac{1}{2} \left( \frac{5}{3} + \frac{5}{4} \right)(1) \]

\[ = \frac{1}{2} \left( \frac{5}{2} + 2 \cdot \frac{5}{2} + 2 \cdot \frac{5}{3} + \frac{5}{4} \right) \]

\[ = \frac{1}{2} \left( \frac{5}{4} \cdot 60 + 60 + 40 + 15 \right) \]

\[ = \frac{175}{24} = 7.2917 \quad \text{(Either of these is fine.)} \]

c.) (4 pts.) approximate the integral using a right-endpoint sum with 3 subintervals (include a picture);

\[ R_3 = \left( \frac{5}{2} \right)(1) + \left( \frac{5}{3} \right)(1) + \left( \frac{5}{4} \right)(1) \]

\[ = \frac{30 + 20 + 15}{12} \]

\[ = \frac{65}{12} = 5.4167 \quad \text{(Either of these is fine.)} \]

d.) (3 pts.) write your answer to part (c.) in \( \Sigma \)-notation.

\[ \sum_{k=2}^{4} \left( \frac{5}{k} \right)(1) = R_3 \]
4.) (15 pts.) Let \( f(x) = |x - 3| \).

a.) (4 pts.) Does \( f \) satisfy all hypotheses of the EVT on \( [2, 5] \)? (In answering this, be sure to state what those hypotheses are.)

Yes: \( f \) is continuous on the closed interval \( [2, 5] \).

b.) (4 pts.) What is the conclusion of the EVT? If you thought the answer in part (a.) was "yes", also compute all points, specific to \( f \), that are guaranteed by the EVT.

\( f \) attains both a \textit{max} and a \textit{min} on \( [2, 5] \).

\[ \begin{align*}
\text{min: } & 0 \quad \text{(at } x = 3) \\
\text{max: } & 2 \quad \text{(at } x = 5) 
\end{align*} \]

c.) (4 pts.) Does \( f \) satisfy all hypotheses of the MVT on \( [2, 5] \)? (In answering this, be sure to state what those hypotheses are.)

\( f \) is continuous on the closed, bounded interval \( [2, 5] \)

BUT:

\( f \) is not differentiable at \( x = 3 \) (because there is a corner in the graph)

\( \rightarrow \) and \( f \) would need to be differentiable on all of \( (2, 5) \)

d.) (3 pts.) What is the conclusion of the MVT? If you thought the answer in part (c.) was "yes", also compute all points, specific to \( f \), that are guaranteed by the MVT.

Conclusion: there exists a number \( c \) between 2 and 5

for which \( f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{f(5) - f(2)}{5 - 2} = \frac{2 - 1}{3} = \frac{1}{3} \)

BUT: The MVT hypotheses are not satisfied, and no such "c" exists.
5.) (15 pts.) Let \( f(x) = \frac{2x + 3}{5x - 4} \).

a.) (5 pts.) Compute \( f'(x) \) using the Quotient Rule.

\[
f'(x) = \frac{(5x - 4)(2) - (2x + 3)(5)}{(5x - 4)^2}
\]

\[
= \frac{10x - 8 - 10x - 15}{(5x - 4)^2}
\]

\[
= \frac{-23}{(5x - 4)^2}
\]

b.) (5 pts.) Compute \( f'(x) \) by first re-writing \( f(x) = (2x + 3)(5x - 4)^{-1} \) and then using the Product Rule.

\[
f'(x) = (5x - 4)^{-1} \cdot (2x + 3) + (2x + 3)[-(5x - 4)^{-2} \cdot 5]
\]

\[
= \frac{2}{5x - 4} - \frac{5(2x + 3)}{(5x - 4)^2}
\]

\[
= \frac{2(5x - 4) - 5(2x + 3)}{(5x - 4)^2}
\]


c.) (5 pts.) Do any necessary algebra to answer this question: are your answers equal to each other? (Should they be?)

\[
= \frac{10x - 8 - 10x - 15}{(5x - 4)^2} = \frac{-23}{(5x - 4)^2}
\]

So parts (a) and (b) simplify to be equal — as they should!
6.) (15 pts.)

a.) (5 pts.) Compute \( h'(x) \) if \( h(x) = \log_2 x + e^x - \sin(\cos x) \).

\[
h'(x) = \frac{1}{\ln 2} \cdot \frac{1}{x} + 0 - \left[ \cos(\cos x) \right] \cdot (-\sin x)
\]

b.) (5 pts.) Solve the IVP \( y' = 3^x \ln 3 - 4, \ y(1) = 5 \).

\[
y = 3^x - 4x + C
\]

\[
5 = 3^1 - 4(1) + C = 3 - 4 + C
\]

\[
5 = -1 + C
\]

\[
C = 6 \quad \Rightarrow \quad y = 3^x - 4x + 6
\]

c.) (5 pts.) Compute the limit \( \lim_{x \to 0} \frac{e^x - x - 1}{3x} \). Be sure to show all necessary steps.

As \( x \to 0 \):

\[
e^x - x - 1 \to e^0 - 0 - 1 = 0
\]

\[
3x \to 3 \cdot 0 = 0
\]

Since the fraction \( \to \frac{0}{0} \), we can use L'Hôpital's Rule.

\[
\lim_{x \to 0} \frac{e^x - x - 1}{3x} = \lim_{x \to 0} \frac{e^x - 1}{3} = \frac{e^0 - 1}{3} = 0
\]
7.) (15 pts.) Given the graph of $f(x)$ below, graph an antiderivative $F(x)$ on the bottom left set of axes, and graph $f'(x)$ on the bottom right set of axes. For the $F(x)$ graph, let $F(0) = -3$. 

Note: To be accurate, the scale for $F(x)$ should extend below the axes shown. We did not focus on scale this semester, however, and it was not part of the grading here.