1. The following matrix A has eigenvalues 4 and 3.

$$A = \begin{bmatrix} 10 & -3 & -6 \\ 2 & 3 & -2 \\ 6 & -3 & -2 \end{bmatrix}$$

1a. Find a basis for the eigenspace of the eigenvalue 4. Show all your work.

1b. Find a basis for the eigenspace of the eigenvalue 3.

1c. Is A diagonalizable? If so, find the required matrices P and D. Otherwise explain why not.

1d. Now you know enough about the eigenvalues to give the characteristic polynomial in factored form. What is it?

2. Let $B = \begin{bmatrix} 2 & 1 & 0 \\ -5 & 4 & 2 \\ -4 & 2 & 4 \end{bmatrix}$; let $\mathbf{w} = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

One of the eigenvalues of B is $\lambda_1 = 4$. Its multiplicity is 1 and a basis for its eigenspace is $\{\mathbf{w}\}$. There is another eigenvalue λ_2 and its multiplicity is 2; one eigenvector corresponding to the eigenvalue λ_2 is $\mathbf{v}$.

2a. Use $\mathbf{v}$ to find λ_2 quickly.

2b. Find a basis for the eigenspace of λ_2.

2c. Is B diagonalizable? If so, find the required matrices P and D. Otherwise explain why not.

3. The points $(0, -5)$, $(1, 6)$, $(4, 3)$ and $(6, 0)$ are certainly not colinear. Find β_0 and β_1 so that $y = \beta_0 + \beta_1 x$ is the least squares line which best fits these points. Show all your work.

4. Let W be the space spanned by $\mathbf{u}_1$ and $\mathbf{u}_2$ where $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 6 \\ 1 \\ 4 \end{bmatrix}$.

4a. What is the projection of $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ on W?

4b. What is the projection of $\mathbf{c} = \begin{bmatrix} 8 \\ 1 \\ 1 \end{bmatrix}$ on W?

4c. Find an orthonormal basis for W.

5. The points $(-3,-4)$, $(2,6)$, and $(4,24)$ *do* all lie on a parabola of the form $\beta_0 + \beta_1 x + \beta_2 x^2$. Set up the system of equations you need to solve in order to find β_0, β_1 and β_2 and proceed to solve it.

6. Let $A = \begin{bmatrix} 1 & 1 & -1 \\ -2 & -1 & 5 \\ 3 & 7 & 9 \end{bmatrix}$, then $[A|I_3]$ is row equivalent to $\left[\begin{array}{ccc|ccc} 1 & 0 & -4 & 0 & -7/11 & -1/11 \\ 0 & 1 & 3 & 0 & 3/11 & 2/11 \\ 0 & 0 & 0 & 1 & 4/11 & -1/11 \end{array}\right]$

6a. Find a basis for $\mathrm{Col}(A)$.

6b. Find a basis for $\mathrm{Nul}(A)$.

6c. What requirements, if any, must b_1, b_2 and b_3 satisfy in order for $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ to be in $\mathrm{Col}(A)$?

6d. Suppose A is the matrix of some linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^3$. Is $\mathbf{b} = \begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix}$ in the range of T? If so, find a vector $\mathbf{v}$ in $\mathbf{R}^3$ such that $T(\mathbf{v}) = \mathbf{b}$.

6e. Let T be as in 7d. Find a nonzero vector $\mathbf{v}$ such that $T(\mathbf{v}) = \mathbf{0}$, or explain what you can't. Then further explain whether T is one-to-one.

7. Suppose $T : \mathbf{R}^2 \to \mathbf{R}^3$ is a linear transformation and $T(\begin{bmatrix} 4 \\ -3 \end{bmatrix}) = \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}$ and $T(\begin{bmatrix} 10 \\ 20 \end{bmatrix}) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Find $T(\begin{bmatrix} 58 \\ 94 \end{bmatrix})$.

8. True or False. Mark a statement TRUE is it is always true and FALSE if it is not always true (ie, if there are counterexamples to the statement).

8a) The only basis of the vector space $\{\mathbf{0}\}$ is $\{\mathbf{0}\}$ itself.

8b) The multiplicity of an eigenvalue of a matrix A is at least as big as the dimension of its eigenspace.

8c) If A is not invertible, then it must have $\lambda = 0$ as an eigenvalue.

8d) If A is 3x3, then it must have at least one real eigenvalue. (Hint: does a cubic polynomial always cross the $x - axis$ at least once?)

8e) If A is 2x2, then it must have at least one real eigenvalue. (Hint: think about roots of quadratic polys)

8f) If A is 3x3 and has three different eigenvalues, A must be diagonalizable.

9. Find the determinants of each of the following matrices, where $Q = \begin{bmatrix} 0 & 5 & 6 & 0 \\ 2 & 12 & 13 & 14 \\ 0 & 3 & 4 & 0 \\ 0 & 4 & 18 & 3 \end{bmatrix}$

9a) Q

9b) QQ

9c) R, where R is obtained from Q by multiplying each row of Q by 10.

9d) Q^{-1}

9e) S where S is obtained from Q by changing every odd number to 0.

Here are some facts you might find useful.

$$\begin{bmatrix} 4 & 11 & 4 \\ 11 & 53 & 18 \end{bmatrix} \text{ is row equivalent to } \begin{bmatrix} 1 & 0 & 2/13 \\ 0 & 1 & 4/13 \end{bmatrix}$$

$$\begin{bmatrix} 7 & -3 & -6 \\ 2 & 0 & -2 \\ 6 & -3 & -5 \end{bmatrix} \text{ is row equivalent to } \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \end{bmatrix}$$